

Advanced modeling and simulation of flow in subsurface reservoirs with fractures and wells for a sustainable industry
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Theoretical stability analysis of mixed finite element model of shale-gas flow with geomechanical effect

Mohamed F. El-Amin^{1,2,*}, Jisheng Kou³, and Shuyu Sun⁴

¹Energy Research Laboratory, College of Engineering, Effat University, 21478 Jeddah, Kingdom of Saudi Arabia

²Department of Mathematics, Faculty of Science, Aswan University, 81528 Aswan, Egypt

³School of Mathematics and Statistics, Hubei Engineering University, Xiaogan, 432000 Hubei, PR China

⁴Division of Physical Sciences and Engineering (PSE), King Abdullah University of Science and Technology (KAUST), Thuwal, 23955-6900 Jeddah, Kingdom of Saudi Arabia

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Abstract. In this work, we introduce a theoretical foundation of the stability analysis of the mixed finite element solution to the problem of shale-gas transport in fractured porous media with geomechanical effects. The differential system was solved numerically by the Mixed Finite Element Method (MFEM). The results include seven lemmas and a theorem with rigorous mathematical proofs. The stability analysis presents the boundedness condition of the MFE solution.

Nomenclature

| | |
|------------|--|
| b_d | Factor of slippage |
| $k_{0,d}$ | Intrinsic permeability |
| ϕ_m | Porosity of the matrix blocks |
| ϕ_f | Porosity of the fractures |
| ρ_m | Mass density of the gas in the matrix blocks |
| ρ_f | Mass density of the gas in the fractures |
| ρ_s | Mass density of the core |
| M_w | Gas molar weight |
| P_L | Langmuir pressure |
| p_m | Gas pressure in matrix blocks |
| p_f | Gas pressure in fractures |
| p_c | Critical pressure |
| p_{fe} | Average fractures pressure around the well |
| V_L | Langmuir volume |
| V_{std} | Mole volume under standard conditions |
| R | General constant of gases |
| T | Temperature |
| T_c | Critical temperature |
| Z | Gas compressibility factor |
| r_w | Well-radius |
| r_e | Drainage-radius |
| n | Set of normal fractures |
| σ_m | Mean stress in matrix blocks |
| σ_f | Mean stress in fractures |

| | |
|-------------|-----------------------------------|
| σ'_m | Effective stress in matrix blocks |
| σ'_f | Effective stress in fractures |
| α_d | Biot's effective parameter |
| L_x | Fracture spacing in x |
| L_y | Fracture spacing in y |
| L_z | Fracture spacing in z |

Superscripts and subscripts

| | |
|-----|------------|
| f | Fractures |
| m | Matrix |
| d | m or f |

1 Introduction

Finite Element Methods (FEMs) are effective numerical techniques for solving the complex engineering problems. FEMs appeared in the middle of the last century and were widely used in solid mechanical applications [1]. The FEM has been extensively investigated and developed in the last five decades [2], and now is used for highly nonlinear problems [3]. However, the standard FEM faced certain instabilities for solving elliptic problem that describe the flow in porous media [4, 5]. On the other hand, the Mixed Finite Element Methods (MFEMs) have succeeded in eliminating such instabilities [6, 7] as it may be extended to higher-order approximations as well as it is a locally

* Corresponding author: momousa@effatuniversity.edu.sa

conservative method [8]. Different physical variables can be treated differently in the MFEM from other variables. For example, the enhanced linear Brezzi-Douglas-Marini (BDM1) space is used for speed approximation, which requires more precision, while the piece-wise constant space is used for pressure approximation. The models of shale-gas transport were mainly developed by adapting the traditional models of flow in fractured porous media. Warren and Root [9] have developed an idealized geometric model to investigate the behavioral flow in fractured porous media. The dual porosity model including stress of fracture was used to investigate the flow of gas in the shale [10, 11], while, the dual-continuum model was used to describe gas flow Kerogen shale [12, 13]. Several versions of the dual-continuum model have been developed to include, for instance, adsorption, heat, deformation and Knudsen diffusion [14–17]. Several authors presented research work in shale gas reservoirs with geomechanical effects, such as Wang *et al.* [18], Lin *et al.* [19], and Yang *et al.* [20]. Wang *et al.* [18] developed a general model including embedded discrete fractures, multiple interacting continua and geomechanics in shale gas reservoirs with multi-scale fractures. In [19], authors provide insights on fracture propagation using reservoir flow simulation integrated with geomechanics in unconventional tight reservoirs. Effects of multicomponent adsorption and geomechanics are investigated in a shale condensate reservoir [20]. Girault *et al.* [21] have introduced *a priori* error estimates for a discretized poro-elastic-elastic system, in which the flow pressure equation is discretized by either a continuous Galerkin scheme or a mixed scheme, while, elastic displacement equations are discretized by a continuous Galerkin scheme. El-Amin *et al.* [22] have used the MFEM with stability analysis to simulate the problem of natural gas transport in a low-permeability reservoir without considering fractures. The authors [23] extended their work to cover fractured porous media and the rock stress-sensitivity with considered stability analysis of the MFEM. In this work, we present a theoretical basis with proofs of the stability analysis of the MFEM (in Ref. [23]) including the necessary lemmas and theorem.

2 Modeling and formulation

In this section, the mathematical model of the problem under consideration is developed. The Dual Porosity Dual Permeability (DPDP) model is employed to describe the gas transport in fractured porous media which consists of matrix blocks and fractures. The absorbed gas and the free gas coexist in the matrix blocks, however, the free gas exists in fractures only. The mass of free gas accumulation is $\phi_m \rho_m$, and the adsorbed gas accumulation (Langmuir isotherm model) is [24, 25],

$$\frac{(1 - \phi_m) \rho_s M_w V_L p_m}{V_{std} (P_L + p_m)}. \quad (1)$$

The mass density of the gas can be written as,

$$\rho_{g,d} = \frac{p_d M_w}{ZRT}, \quad d = m, f. \quad (2)$$

The gas compressibility factor, Z which is given by Peng-Robinson equation of state [26],

$$Z^3 - (1 - B)Z^2 + (A - 3B^3 - 2B)Z - (AB - B^2 - B^3) = 0, \quad (3)$$

$$A = \frac{a_T p}{R^2 T^2}, \quad B = \frac{b_T p}{RT}, \quad (4)$$

$$a_T = 0.45724 \frac{R^2 T_c^2}{p_c}, \quad b_T = 0.0778 \frac{RT_c}{p_c}. \quad (5)$$

In low-permeability formations, Klinkenberg effect, which is described by the apparent permeability, takes place and written as,

$$k_{app,d} = k_{0,d} \left(1 + \frac{b_d}{p_d} \right), \quad d = m, f. \quad (6)$$

The DPDP model of gas transport in fracture shale strata is represented as,

$$f_1(p_m) \frac{\partial p_m}{\partial t} - \nabla \cdot \left[\frac{\rho_{g,m}(p_m)}{\mu} k_{0,m} \left(1 + \frac{b_m}{p_m} \right) \nabla p_m \right] = -S(p_m, p_f), \quad (7)$$

and

$$f_2(p_f) \frac{\partial p_f}{\partial t} - \nabla \cdot \left[\frac{\rho_{g,f}(p_f)}{\mu} k_{0,f} \left(1 + \frac{b_f}{p_f} \right) \nabla p_f \right] = S(p_m, p_f) - Q(p_f), \quad (8)$$

where

$$f_1(p_m) = \frac{M_w}{ZRT} [\phi_m(p_m) + \phi'_m(p_m) p_m] + \frac{M_w V_L \rho_s}{V_{std}} \left(\frac{P_L (1 - \phi_m(p_m))}{(P_L - p_m)^2} - \frac{\phi'_m(p_m)}{P_L + p_m} \right), \quad (9)$$

and

$$f_2(p_f) = \frac{M_w}{ZRT} [\phi_f(p_f) + \phi'_f(p_f) p_f], \quad (10)$$

such that ϕ_m and ϕ_f is the matrix and fractures porosities, respectively,

$$\phi'_d(p_d) = \frac{d\phi_d}{dp_d}, \quad d = m, f. \quad (11)$$

Moreover, based on the following definitions, b_m and b_f are treated as constants [17],

$$b_m = \sqrt{\frac{8\pi RT}{M_w}} \frac{1}{r} \left(\frac{2}{\alpha} - 0.995 \right) \mu_g, \quad (12)$$

and

$$b_f = \sqrt{\frac{\pi RT \phi_{0,f}}{M_w k_{0,f}}} \mu_g. \quad (13)$$

The Knudsen diffusion coefficient is defined as,

$$D_{kf} = \sqrt{\frac{\pi RTk_{0,f}\phi_{0,f}}{M_w}}. \quad (14)$$

Here, $S(p_m, p_f)$ is the transfer parameter which is connecting of the matrix-fracture domains, and defined as,

$$S(p_m, p_f) = \frac{\sigma\rho_{g,m}k_m}{\mu}(p_m - p_f). \quad (15)$$

One may provide the definition of the source of the production well by,

$$Q(p_f) = \frac{\theta\rho_{g,f}k_f}{\mu \ln \frac{r_e}{r_w}}(p_{fe} - p_w), \quad (16)$$

$$\theta = \begin{cases} \theta = 2\pi & \text{if the production well at the center of the reservoir} \\ \theta = \frac{\pi}{2} & \text{if the production well at the corner of the reservoir} \end{cases}$$

Given the constant, r_e , the drainage-radius is represented as,

$$r_e = 0.14\sqrt{(\Delta x)^2 + (\Delta y)^2}. \quad (17)$$

The coefficient of crossflow between the matrix and fracture domains is defined as [9],

$$\sigma = \frac{2n(n+1)}{l^2}, \quad (18)$$

such that l is given by,

$$l = \frac{3L_xL_yL_z}{L_xL_y + L_yL_z + L_xL_z}. \quad (19)$$

2.1 Geomechanical effects

Invoking the rock stress-sensitivity [27], the porosity can be expressed as a function of the mean effective-stress,

$$\phi_d(\sigma'_d(p_d)) = \phi_{r,d} + (\phi_{0,d} - \phi_{r,d})\exp(-a\sigma'_d), \quad d = m, f, \quad (20)$$

$$\sigma'_d = \sigma_d + \alpha_d p_d, \quad d = m, f. \quad (21)$$

From (20) and (21), we can obtain,

$$\begin{aligned} \phi_d(p_d) &= \phi_{r,d} + C_{1,d}\exp(-ap_d), \quad C_{1,d} \\ &= (\phi_{0,d} - \phi_{r,d})\exp(-a\sigma_d), \quad d = m, f. \end{aligned} \quad (22)$$

As the porosity increases the difference $(\phi_{0,d} - \phi_{r,d})$ becomes positive and *vice versa*. Also, the coefficient $C_{1,d}$ becomes small based on the change in the porosity, and it becomes positive if porosity increases and negative if porosity decreases.

2.2 Initial and boundary conditions

The initial conditions can be represented by,

$$p_m(\cdot, 0) = p_f(\cdot, 0) = p_0 \quad \text{in} \quad \Omega_m \cup \Omega_f. \quad (23)$$

The pressure on the production well (boundary condition) is given as,

$$p_f(\cdot, t) = p_w \quad \text{on} \quad \Gamma_D^f \times (0, T). \quad (24)$$

The no-flow boundary conditions on matrix are given as,

$$u_m \cdot n = 0 \quad \text{on} \quad \Gamma_N^m \cup \Gamma_N^f \times (0, T), \quad (25)$$

while the no-flow boundary conditions for fracture domain are,

$$u_f \cdot n = 0 \quad \text{on} \quad \Gamma_N^f \times (0, T). \quad (26)$$

3 MFEM spaces

The MFEM contains two spaces for a scalar variable and its flux. The MFEM was developed to approximate both variables simultaneously and to give a higher order approximation for the flux. A compatibility condition must hold for the two spaces to insure stability, consistency and convergence of the mixed method and by considering more constraints to the numerical discretization.

Now, let us define the inner product in Ω as,

$$(f, g)_\Omega = \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{V} \quad \forall f, g : \Omega \rightarrow \mathbb{R}, \quad (27)$$

and the inner product on $\partial\Omega$ as,

$$\langle f, g \rangle_{\partial\Omega} = \int_{\partial\Omega} fg dS \quad \forall f, g : \partial\Omega \rightarrow \mathbb{R}. \quad (28)$$

Given,

$$\mathbb{L}^2(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : \int_\Omega f^2 dx < +\infty\}, \quad (29)$$

is the largest Hilbert space, such that $(D^{-1}\mathbf{u}, w)$ and $(p, \nabla \cdot w)$ are well defined, therefore, $p, \nabla \cdot w \in \mathbb{L}^2(\Omega)$. It is needed that $p, \varphi \in \mathbb{L}^2(\Omega)$ and $\mathbf{u}, w \in \mathbb{H}(\text{div}, \Omega)$. The Hilbert space with norm given by,

$$\mathbb{H}(\text{div}, \Omega) = \{\mathbf{u} : \nabla \cdot \mathbf{u} \in \mathbb{L}^2(\Omega)\}, \quad (30)$$

$$\|\mathbf{u}\|_{\mathbb{H}(\text{div}, \Omega)}^2 = \|\mathbf{u}\|^2 + \|\nabla \cdot \mathbf{u}\|^2. \quad (31)$$

The seeking solution is,

$$(p, \mathbf{u}) \in \mathbb{L}^2(\Omega) \times \mathbb{H}(\text{div}, \Omega), \quad (32)$$

such that p and \mathbf{u} are smooth. For the approximate numerical solution, the two spaces become,

$$W_h \subset \mathbb{L}^2(\Omega), \quad \dim(W_h) < +\infty \quad (33)$$

and,

$$V_h \subset \mathbb{H}(\text{div}, \Omega), \quad \dim(V_h) < +\infty. \quad (34)$$

Thus, the normal component $\mathbf{u} \cdot \mathbf{n}$ is continuous across the inter-element boundaries.

Moreover, it is useful to present the \mathbf{RT}_r elements that are designed to approximate $\mathbb{H}(\text{div}, \Omega)$ [7], which satisfies,

$$W_h = \{w \in \mathbb{L}^2(\Omega) : w|_E \in \mathbb{P}_0(E), E \in \mathcal{E}_h\}, \quad (35)$$

and,

$$V_h = \{\mathbf{u} \in \mathbb{H}(\text{div}, \Omega) : \mathbf{u}|_E \in \mathbb{P}_1(E), E \in \mathcal{E}_h\}, \quad (36)$$

where w is discontinuous piecewise constant and \mathbf{u} is piecewise linear.

4 Mixed finite element approximation

Let Ω_m and Ω_f are, respectively, the matrix and fracture in a polygonal/polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ (on which we define the standard $L^2(\Omega)$ space such that $\mathbb{L}^2(\Omega) \equiv (L^2(\Omega))^d$), with the boundaries, $\partial\Omega_m = \Gamma_D^m \cup \Gamma_N^m$ and $\partial\Omega_f = \Gamma_D^f \cup \Gamma_N^f$. One may write the above dual-porosity model in the following general form,

$$f_1(p_m) \frac{\partial p_m}{\partial t} + \nabla \cdot \mathbf{u}_m = -S(p_m, p_f) \quad \text{in } \Omega_m \times (0, T), \quad (37)$$

$$\mathbf{D}_m(p_m)^{-1} \mathbf{u}_m = -\nabla p_m \quad \text{in } \Omega_m \times (0, T), \quad (38)$$

$$f_2(p_f) \frac{\partial p_f}{\partial t} + \nabla \cdot \mathbf{u}_f = S(p_m, p_f) - Q(p_f) \quad \text{in } \Omega_f \times (0, T), \quad (39)$$

$$\mathbf{D}_f(p_f)^{-1} \mathbf{u}_f = -\nabla p_f \quad \text{in } \Omega_f \times (0, T), \quad (40)$$

where

$$\mathbf{D}_m(p_m) = \frac{\rho(p_m)}{\mu} k_{0,m} \left(1 + \frac{b_m}{p_m}\right), \quad (41)$$

and

$$\mathbf{D}_f(p_f) = \frac{\rho(p_f)}{\mu} k_{0,f} \left(1 + \frac{b_f}{p_f}\right). \quad (42)$$

The functions $\mathbf{D}_m(p_m)^{-1}$ and $\mathbf{D}_f(p_f)^{-1}$ are moved to the left hand side to avoid discontinuity when we integrate ∇p_m and ∇p_f by parts. Selecting any $\varphi \in W_h$ and $\omega \in V_h$, the mixed finite element weak formulation can be written in the following form,

$$\left(f_1(p_m) \frac{\partial p_m}{\partial t}, \varphi\right) + (\nabla \cdot \mathbf{u}_m, \varphi) + (S(p_m, p_f), \varphi) = 0, \quad (43)$$

$$(\mathbf{D}_m(p_m)^{-1} \mathbf{u}_m, \omega) = (p_m, \nabla \cdot \omega), \quad (44)$$

$$\left(f_2(p_f) \frac{\partial p_f}{\partial t}, \varphi\right) + (\nabla \cdot \mathbf{u}_f, \varphi) - (S(p_m, p_f), \varphi) = -(Q(p_f), \varphi), \quad (45)$$

$$(\mathbf{D}_f(p_f)^{-1} \mathbf{u}_f, \omega) = (p_f, \nabla \cdot \omega) - \langle p_w, \omega \rangle_{\Gamma_f^D}. \quad (46)$$

Now, let the approximating subspace duality $V_h \subset H(\Omega; \text{div})$ and $W_h \subset L^2(\Omega)$ be the r -th order ($r \geq 0$) Raviart–Thomas space (\mathbf{RT}_r) on the partition \mathcal{T}_h . The mixed finite element formulations are stated as below: find $p_m^h, p_f^h \in W_h$ and $\mathbf{u}_m^h, \mathbf{u}_f^h \in V_h$ such that,

$$\left(f_1(p_m^h) \frac{\partial p_m^h}{\partial t}, \varphi\right) + (\nabla \cdot \mathbf{u}_m^h, \varphi) + (S(p_m^h, p_f^h), \varphi) = 0, \quad (47)$$

$$(\mathbf{D}_m(p_m^h)^{-1} \mathbf{u}_m^h, \omega) = (p_m^h, \nabla \cdot \omega), \quad (48)$$

$$\left(f_2(p_f^h) \frac{\partial p_f^h}{\partial t}, \varphi\right) + (\nabla \cdot \mathbf{u}_f^h, \varphi) - (S(p_m^h, p_f^h), \varphi) = -(Q(p_f^h), \varphi), \quad (49)$$

$$(\mathbf{D}_f(p_f^h)^{-1} \mathbf{u}_f^h, \omega) = (p_f^h, \nabla \cdot \omega) - \langle p_w, \omega \rangle_{\Gamma_f^D}, \quad (50)$$

for any $\varphi \in W_h$ and $\omega \in V_h$.

In order to obtain an explicit formulation for the flux, we employ a quadrature rule along with the MFE method to [8]. The total time interval $[0, T]$ is divided into N_T time steps with length $\Delta t^n = t^{n+1} - t^n$. The superscript $n+1$ denotes for the current time step, while n denotes for the previous one. We use a backward Euler semi-implicit discretization for the time derivative terms. The following scheme has been developed,

$$\left(f_1(p_m^{h,n}) \frac{p_m^{h,n+1} - p_m^{h,n}}{\Delta t}, \varphi\right) + (\nabla \cdot \mathbf{u}_m^{h,n+1}, \varphi) + (S(p_m^{h,n+1}, p_f^{h,n}), \varphi) = 0, \quad (51)$$

$$(\mathbf{D}_m(p_m^{h,n})^{-1} \mathbf{u}_m^{h,n+1}, \omega) = (p_m^{h,n+1}, \nabla \cdot \omega), \quad (52)$$

$$\left(f_2(p_f^{h,n}) \frac{p_f^{h,n+1} - p_f^{h,n}}{\Delta t}, \varphi\right) + (\nabla \cdot \mathbf{u}_f^{h,n+1}, \varphi) - (S(p_m^{h,n+1}, p_f^{h,n+1}), \varphi) = -(Q(p_f^{h,n+1}), \varphi), \quad (53)$$

$$(\mathbf{D}_f(p_f^{h,n})^{-1} \mathbf{u}_f^{h,n+1}, \omega) = (p_f^{h,n+1}, \nabla \cdot \omega) - \langle p_w, \omega \rangle_{\Gamma_f^D}. \quad (54)$$

Given $p_m^{h,n}$ and $p_f^{h,n}$, the numerical procedure to calculate pressure and velocity is presented here,

1. Update the thermodynamical variables explicitly.
2. Solve the equations (51) and (52) to obtain $p_m^{h,n+1}$ and $\mathbf{u}_m^{h,n+1}$.
3. Solve the equations (53) and (54) to obtain $p_f^{h,n+1}$ and $\mathbf{u}_f^{h,n+1}$.

5 Stability analysis

In this section, we carry out the stability analysis of the proposed MFE method, which ensures that the discrete solutions are bounded in a physically reasonable range. The key issue encountered in the stability analysis is the nonlinearity of the matrix and fracture pressures. In order to resolve this issue, we need to define some auxiliary functions and analyze their boundedness. Now let us define the following functions,

$$F_1(p_d^h) = \int_0^{p_d^h} f_1(p_d^h) dp_d^h, \quad G_1(p_d^h) = \int_0^{p_d^h} F_1(p_d^h) dp_d^h, \quad (55)$$

$$F_2(p_d^h) = \int_0^{p_d^h} f_2(p_d^h) dp_d^h, \quad G_2(p_d^h) = \int_0^{p_d^h} F_2(p_d^h) dp_d^h. \quad (56)$$

Therefore, one may

$$\begin{aligned} F_1(p_m^h) &= (\phi_{r,m} + C_{1,m}) \frac{M_w}{ZRT} p_m^h e^{-ap_m^h} \\ &+ \frac{M_w V_L \rho_s}{V_{std}} \left[C_{1,m} \left(a e^{aP_L} (1 + P_L) Ei(-x) \Big|_{aP_L}^{a(P_L+p_m^h)} \right. \right. \\ &\left. \left. + \frac{P_L e^{-ap_m^h}}{P_L + p_m^h} - 1 \right) + (1 - \phi_{r,m}) \frac{p_m^h}{P_L + p_m^h} \right], \quad (57) \end{aligned}$$

$$\begin{aligned} G_1(p_m^h) &= (\phi_{r,m} + C_{1,m}) \frac{M_w}{a^2 ZRT} \left[1 - (ap_m^h + 1) e^{-ap_m^h} \right] \\ &+ \frac{M_w V_L \rho_s}{V_{std}} \left[C_{1,m} \left(a e^{aP_L} (1 + P_L) \left((P_L + p_m^h) Ei(-x) \Big|_{aP_L}^{a(P_L+p_m^h)} \right. \right. \right. \\ &\left. \left. - \frac{1}{a} (1 - e^{-ap_m^h}) \right) P_L e^{aP_L} + Ei(-x) \Big|_{aP_L}^{a(P_L+p_m^h)} \right) \\ &\left. + (1 - \phi_{r,m} - C_{1,m}) p_m^h + (1 - \phi_{r,m}) P_L \ln \left(\frac{P_L}{P_L + p_m^h} \right) \right], \quad (58) \end{aligned}$$

$$F_2(p_f^h) = (\phi_{r,f} + C_{1,f}) \frac{M_w}{ZRT} p_f^h e^{-ap_f^h}, \quad (59)$$

and

$$G_2(p_f^h) = (\phi_{r,f} + C_{1,f}) \frac{M_w}{a^2 ZRT} \left[1 - (ap_f^h + 1) e^{-ap_f^h} \right]. \quad (60)$$

$Ei(x)$ is a special function called the exponent integral function. As stated above in the model formulation that the coefficient $C_{1,d}$, $d = m, f$ is positive in the case of increasing porosity and negative in the case of decreasing porosity.

Lemma 5.1 For the case of increasing matrix porosity, $C_{1,m} > 0$, and sufficiently large p_m^h , there exists a positive constant,

$$\begin{aligned} \gamma_1 &= (\phi_{r,m} + C_{1,m}) C_4^l \frac{M_w}{ZRT} + \frac{M_w V_L \rho_s}{V_{std}} \\ &(1 - \phi_{r,m}) C_3^l, \quad (61) \end{aligned}$$

such that,

$$(F_1(p_m^h), p_m^h) \geq \gamma_1 \|p_m^h\|^2. \quad (62)$$

Proof. Note that the value of the exponent integral function $Ei(-x)$ is negative and (small for big values of x), therefore the quantity, $Ei(-x) \Big|_{aP_L}^{a(P_L+p_m^h)}$ has always a positive value which has lower and upper bounds, namely,

$$C_{2,m}^u \geq Ei(-x) \Big|_{aP_L}^{a(P_L+p_m^h)} \geq C_{2,m}^l, \quad (63)$$

where $C_{2,m}^l, C_{2,m}^u > 0$. On the other hand, in shale reservoir, it is well known that the initial pressure has the maximum value, *i.e.*, $\max(p_m^h) = P_0$, while, the pressure of the well has the minimum value, *i.e.*, $\min(p_m^h) = P_w$. Therefore, we have,

$$C_3^u = \frac{1}{P_L + p_w} \geq \frac{1}{P_L + p_m^h} \geq \frac{1}{P_L + p_0} = C_3^l, \quad (64)$$

$$C_4^u = e^{-ap_w} \geq e^{-ap_m^h} \geq e^{-ap_0} = C_4^l, \quad (65)$$

and,

$$C_5^u = \frac{e^{-ap_w}}{P_L + p_w} \geq \frac{e^{-ap_m^h}}{P_L + p_m^h} \geq \frac{e^{-ap_0}}{P_L + p_0} = C_5^l, \quad (66)$$

where $C_3^l, C_3^u, C_4^l, C_4^u, C_5^l, C_5^u \geq 0$. Therefore, for the case of increasing matrix porosity, $C_{1,m} > 0$, and sufficiently large p_m^h , and holding (63)–(66), the following coefficient is positive, *i.e.*,

$$a e^{aP_L} (1 + P_L) Ei(-x) \Big|_{aP_L}^{a(P_L+p_m^h)} + \frac{P_L e^{-ap_m^h}}{P_L + p_m^h} - 1 >, \quad (67)$$

$$a e^{aP_L} (1 + P_L) C_{2,m}^l + P_L C_5^l - 1 = C_{3,m} > 0. \quad (68)$$

Therefore,

$$\begin{aligned} (F_1(p_m^h), p_m^h) &\geq (\phi_{r,m} + C_{1,m}) C_4^l \frac{M_w}{ZRT} (p_m^h, p_m^h) \\ &+ \frac{M_w V_L \rho_s}{V_{std}} [C_{1,m} C_{3,m} + (1 - \phi_{r,m}) C_3^l (p_m^h, p_m^h)] \\ &\geq (\phi_{r,m} + C_{1,m}) C_4^l \frac{M_w}{ZRT} (p_m^h, p_m^h) + \frac{M_w V_L \rho_s}{V_{std}} (1 - \phi_{r,m}) C_3^l (p_m^h, p_m^h) \\ &= \left[(\phi_{r,m} + C_{1,m}) C_4^l \frac{M_w}{ZRT} + \frac{M_w V_L \rho_s}{V_{std}} (1 - \phi_{r,m}) C_3^l \right] \\ &\|p_m^h\|^2 = \gamma_1 \|p_m^h\|^2, \quad (69) \end{aligned}$$

and this completes Lemma 5.1 proof.

Lemma 5.2 For the case of decreasing matrix porosity, $C_{1,m} < 0$, assuming,

$$ae^{aP_L}(1 + P_L)Ei(-x)|_{aP_L}^{a(P_L+p_m^h)} + \frac{P_L e^{-ap_m^h}}{P_L + p_m^h} - 1 < 0, \quad (70)$$

there exists a positive constant γ_1 , such that,

$$(F_1(p_m^h), p_m^h) \geq \gamma_1 \|p_m^h\|^2. \quad (71)$$

Proof. It is clear that, $C_{1,m} < \phi_{r,m}$, so, we always have, $\phi_{r,m} - C_{1,m} > 0$. Holding the assumption (70), one can easily prove this lemma in a similar way to Lemma 5.1.

Lemma 5.3 For the case of increasing matrix porosity, $C_{1,m} > 0$, and for sufficiently large p_m^h , there exist two positive constants,

$$\begin{aligned} \gamma_2 &= \frac{M_w}{a^2 ZRT} (\phi_{r,m} + C_{1,m}) (1 - C_{4,m}^u) \\ &+ \frac{M_w V_L \rho_s}{V_{std}} \left[C_{1,m} (ae^{aP_L}(1 + P_L)P_L C_{2,m}^u + P_L e^{aP_L} C_{2,m}^u) \right. \\ &\left. + (1 - \phi_{r,m})P_L \ln(P_L C_3^u) \right], \quad (72) \end{aligned}$$

and

$$\gamma_2' = \frac{M_w V_L \rho_s}{V_{std}} \left[C_{1,m} a e^{aP_L} (1 + P_L) C_{2,m}^u + (1 - \phi_{r,m} - C_{1,m}) \right], \quad (73)$$

such that,

$$(G_1(p_m^h), 1) \leq \gamma_2 |\Omega| + \gamma_2' \|p_m^h\|. \quad (74)$$

Proof. Holding (63)–(66) with $C_{4,m}^u = \frac{ap_m^h+1}{e^{ap_m^h}} < 1$ for sufficiently large p_m^h , and integrate (58) over Ω_m for the case of increasing porosity, $C_{1,m} > 0$, it is easy to prove (74), which completes the lemma proof.

Lemma 5.4 For the case of decreasing matrix porosity, $C_{1,m} < 0$, with assuming that,

$$\begin{aligned} C_{1,m} (ae^{aP_L}(1 + P_L)P_L C_{2,m}^u + P_L e^{aP_L} C_{2,m}^u) \\ + (1 - \phi_{r,m})P_L \ln(P_L C_3^u) \geq 0, \quad (75) \end{aligned}$$

and

$$C_{1,m} a e^{aP_L} (1 + P_L) C_{2,m}^u + (1 - \phi_{r,m} - C_{1,m}) \geq 0, \quad (76)$$

one can prove,

$$(G_1(p_m^h), 1) \leq \gamma_2 |\Omega| + \gamma_2' \|p_m^h\|. \quad (77)$$

Proof. It is clear that, $C_{1,m} < \phi_{r,m}$, so, we always have, $\phi_{r,m} - C_{1,m} > 0$. Holding the assumptions (75) and (76), therefore we find, $\gamma_2 > 0$, $\gamma_2' > 0$, and integrate (58) over Ω_m for the case of decreasing porosity, $C_{1,m} < 0$, which proves (74), and this completes the proof.

Lemma 5.5 For both cases of increasing or decreasing fracture porosity, namely, $C_{1,f} > 0$ or $C_{1,f} < 0$, and sufficiently large p_f^h , there exists a positive constant,

$$\gamma_3 = (\phi_{r,f} + C_{1,f}) \frac{M_w}{ZRT} C_4^l, \quad (78)$$

such that,

$$(F_2(p_f^h), p_f^h) \geq \gamma_3 \|p_f^h\|^2. \quad (79)$$

Proof. Again, we have,

$$C_4^u = e^{-ap_w} \geq e^{-ap_f^h} \geq e^{-ap_0} = C_4^l, \quad (80)$$

where $C_4^l, C_4^u > 0$. Therefore, for the case of increasing fracture porosity, $C_{1,f} > 0$, and sufficiently large p_f^h , Therefore,

$$(F_2(p_f^h), p_f^h) \geq (\phi_{r,f} + C_{1,f}) \frac{M_w}{ZRT} C_4^l (p_f^h, p_f^h) = \gamma_3 \|p_f^h\|^2. \quad (81)$$

As stated above, for the case of the decreasing porosity, $C_{1,f} < 0$, the coefficient $(\phi_{r,f} + C_{1,f})$ remains positive, then the above inequality holds. This completes the lemma proof.

Lemma 5.6 For both cases of increasing or decreasing fracture porosity, namely, $C_{1,f} > 0$ or $C_{1,f} < 0$, and sufficiently large p_f^h , there exists a positive constant,

$$\gamma_4 = \frac{M_w}{a^2 ZRT} (\phi_{r,m} + C_{1,m}) (1 - C_{4,m}^u), \quad (82)$$

such that,

$$(G_2(p_f^h), 1) \leq \gamma_4 |\Omega|. \quad (83)$$

Proof. Similar to the previous proofs.

Lemma 5.7 Assuming that,

$$\begin{aligned} 0 < \rho_{g*} \leq \rho_g \leq \rho_g^*, \quad 0 < k_{f*} \leq k_f \leq k_f^*, \\ 0 < \mu_{g*} \leq \mu_g \leq \mu_g^*, \end{aligned} \quad (84)$$

then, $Q(p_f^h)$ is bounded.

Proof. Since p_{fe} is bounded by p_w and p_0 , i.e., $p_w \leq p_{fe} \leq p_0$, then, $0 < p_{fe} - p_w \leq p_0 - p_w = C_p$, such that C_p is positive number. Holding the conditions (84), one may find that,

$$0 < \frac{k_f(p_f^h) \rho_g(p_f^h) [p_{fe} - p_w]}{\mu_g} \leq \frac{k_f^* \rho_g^* C_p}{\mu_g^*} = C_Q > 0. \quad (85)$$

Theorem 8 Holding the results in the above lemmas (5.1–5.6), namely,

$$(F_1(p_m^h), p_m^h) \geq \gamma_1 \|p_m^h\|^2, \quad (G_1(p_m^h), 1) \leq \gamma_2 |\Omega| + \gamma_2' \|p_m^h\|, \quad (86)$$

and

$$(F_2(p_f^h), p_f^h) \geq \gamma_3 \|p_f^h\|^2, \quad (G_2(p_f^h), 1) \leq \gamma_4 \|1\|, \quad (87)$$

with assuming that,

$$\gamma_1 \|p_m^h\|^2 \geq \gamma_2 |\Omega| + \gamma_2' \|p_m^h\|, \quad (88)$$

and

$$\gamma_3 \|p_f^h\|^2 \geq \gamma_4 |\Omega|. \quad (89)$$

Moreover, suppose that $p_{fe}, p_w \in L^2(0, T; L^2(\Omega))$. Then,

$$\begin{aligned} & \|p_m^h\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|D_m(p_m^h)^{-1/2} u_m^h\|_{L^2(0, T; L^2(\Omega))}^2 \\ & + \|p_f^h\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|D_f(p_f^h)^{-1/2} u_f^h\|_{L^2(0, T; L^2(\Omega))}^2 \\ & + \|p_m^h - p_f^h\|_{L^2(0, T; L^2(\Omega))}^2 \leq C(F_1(p_0), p_0) + C(F_2(p_0), p_0) \\ & + C(G_1(p_0), 1) + C(G_2(p_0), 1) + C(\|p_{fe}\|_{L^2(0, T; L^2(\Omega))}^2) \\ & + \|p_w\|_{L^2(0, T; L^2(\Omega))}^2. \end{aligned} \quad (90)$$

Proof. Let $\varphi = p_m^h$ in (47) and $\omega = u_m^h$ in (48), and sum the two equations, we get,

$$\left(f_1(p_m^h) \frac{\partial p_m^h}{\partial t}, p_m^h \right) + \|D_m(p_m^h)^{-1/2} u_m^h\|^2 + (\mathcal{S}(p_m^h, p_f^h), p_m^h) = 0. \quad (91)$$

It follows from the definitions of F_1 and G_1 that,

$$\begin{aligned} \left(f_1(p_m^h) \frac{\partial p_m^h}{\partial t}, p_m^h \right) &= \frac{\partial}{\partial t} (F_1(p_m^h), p_m^h) - \left(F_1(p_m^h), \frac{\partial p_m^h}{\partial t} \right) \\ &= \frac{\partial}{\partial t} (F_1(p_m^h), p_m^h) - \left(\frac{\partial G_1(p_m^h)}{\partial t}, 1 \right). \end{aligned} \quad (92)$$

Integrating (92) from 0 to t ($0 < t \leq T$) yields,

$$\begin{aligned} & \gamma_1 \|p_m^h\|^2(t) - \gamma_2 |\Omega|(t) - \gamma_2' \|p_m^h\|(t) \\ & + \int_0^t \|D_m(p_m^h)^{-1/2} u_m^h\|^2 + \int_0^t (\mathcal{S}(p_m^h, p_f^h), p_m^h) \\ & \leq (F_1(p_0), p_0) - (G_1(p_0), 1). \end{aligned} \quad (93)$$

It is similar to obtain,

$$\begin{aligned} & \gamma_3 \|p_f^h\|^2(t) - \gamma_4 |\Omega|(t) + \int_0^t \|D_f(p_f^h)^{-1/2} u_f^h\|^2 \\ & + \int_0^t \langle p_w, u_w \rangle_{\Gamma_f^p} - \int_0^t (\mathcal{S}(p_m^h, p_f^h), p_f^h) \\ & \leq (F_2(p_0), p_0) - (G_2(p_0), 1) - \int_0^t (Q(p_f^h), p_f^h). \end{aligned} \quad (94)$$

There exists a positive constant α_0 such that,

$$\left(\mathcal{S}(p_m^h, p_f^h), p_m^h \right) - \left(\mathcal{S}(p_m^h, p_f^h), p_f^h \right) \geq \alpha_0 \|p_m^h - p_f^h\|^2. \quad (95)$$

Thus, it is obtained from, (93)–(95) that,

$$\begin{aligned} & \gamma_1 \|p_m^h\|^2(t) - \gamma_2 |\Omega|(t) - \gamma_2' \|p_m^h\|(t) \\ & + \int_0^t \|D_m(p_m^h)^{-1/2} u_m^h\|^2 + \gamma_3 \|p_f^h\|^2(t) - \gamma_4 |\Omega|(t) \\ & + \int_0^t \|D_f(p_f^h)^{-1/2} u_f^h\|^2 + \alpha_0 \int_0^t \|p_m^h - p_f^h\|^2 \\ & \leq (F_1(p_0), p_0) + (F_2(p_0), p_0) - (G_1(p_0), 1) \\ & - (G_2(p_0), 1) + C \int_0^t (\|p_{fe}\|^2 + \|p_w\|^2 + \|p_f^h\|^2). \end{aligned} \quad (96)$$

Finally, (90) is obtained by Gronwall's lemma.

Remark 5.1. In Lemma 5.6, if we do not use the coefficient, $C_{4,m}^u = \frac{ap_m^h + 1}{e^{ap_m^h}} < 1$, and just use the coefficient, $C_m^u = e^{-ap_w}$, then the coefficient of the term, ap_m^h in $C_{4,m}^u$ will be moved from γ_2 into γ_2' . Thus, the definitions of both γ_2 and γ_2' will be slightly changed without loss of the stability concept.

6 Conclusion

In this work, stability analysis of the MFE solution of the gas transport in a low-permeability fractured reservoir with stress-sensitivity effect has been considered. The dual-porosity dual permeability model with the slippage effect and the apparent permeability have been used to describe the flow in a low-permeability fractured porous media. Stability analysis of the MFEM is presented theoretically and numerically. We proved a seven lemmas and a theorem on the stability of the MFEM. Stability conditions are stated and estimated. Variation in the porosity is correlated to the stress-sensitivity effect and depends on the values of the corresponding physical parameters. For example, the coefficient $C_{1,d}$ is relatively small depending on the variation of the porosity. It has a positive value in the case of porosity increasing while it has a negative value in the case of porosity decreasing. Lemmas 5.1 and 5.3 discuss the positive case, while Lemmas 5.2 and 5.4 discuss the positive case. Lemmas 5.5 and 5.6 discuss the two cases.

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