CONDITIONS FOR THE OCCURRENCE OF DECOUPLING PLANES IN ANISOTROPIC ELASTIC MATERIALS

K. HELBIG[1]

CONDITIONS POUR LA PRÉSENCE DE PLANS DE DÉCOUPLAGE DANS LES MATÉRIAUX ÉLASTIQUES ANISOTROPES

Les plans de symétrie sont souvent identifiés par l'existence d'une polarisation pure perpendiculaire à ces plans. Cependant, ce type de polarisation peut apparaître sans que le plan soit un plan de symétrie. Les plans qui présentent une polarisation pure suivant leurs normales sont appelés « plans de découplage », car le système de trois équations linéaires couplées par les cosinus directeurs du vecteur de polarisation se découple en une seule équation relative à la polarisation normale au plan et en deux équations couplées relatives aux polarisations dans le plan. C'est un plan de symétrie uniquement dans le cas où la direction perpendiculaire à un « plan de découplage » est une « direction longitudinale » (c'est-à-dire, si dans cette direction, il y a des ondes P et des ondes S pures). Sans la présence de la direction purement longitudinale associée, un plan de découplage « brut » pourrait être interprété faussement comme un plan de symétrie.

Le plan perpendiculaire à la direction i est un plan de découplage si, en notation à quatre indices, toutes les rigidités avec un seul indice i disparaissent ; la direction i est une direction longitudinale si toutes les rigidités avec trois indices i disparaissent. Dans les milieux de symétrie orthorhombique ou supérieure, toutes les rigidités ayant un indice simple ou triple disparaissent ; par conséquent, les plans de découplage peuvent apparaître uniquement dans les milieux ayant une symétrie monoclinique ou triclinique.

Dans la symétrie triclinique, deux plans de découplage brut perpendiculaires peuvent apparaître. Des plans de découplage qui se croisent avec un angle oblique sont possibles si les rigidités satisfont un certain nombre de contraintes.

CONDITIONS FOR THE OCCURRENCE OF DECOUPLING PLANES IN ANISOTROPIC ELASTIC MATERIALS

Planes of symmetry are often identified by the existence of pure cross-plane polarization. However, this type of polarization can occur without the plane being a plane of symmetry. Planes that support cross-plane polarization are called "decoupling planes", since the system of three coupled linear equations in the direction cosines of the polarization vector decouples into a single

(1) Kiebitzrain 84,
D 30657 Hannover

cross-plane equation and a coupled pair of in-plane equations. Only if the direction perpendicular to a decoupling plane is a "longitudinal direction" (i.e., if in the direction there are pure P- and S-waves), it is a plane of symmetry. Without the observation of the associated longitudinal direction, a "raw" decoupling plane might be mis-interpreted as a plane of symmetry.

The plane perpendicular to the i-direction is a decoupling plane if in four-subscript notation all stiffnesses with a single subscript i vanish; the i-direction is a longitudinal di-rection if all stiffnesses with three subscripts i vanish. In media of orthorhombic or higher symmetry all stiffnesses with any single or triple subscript vanish; therefore raw decoupling planes can occur only in media of monoclinic or triclinic symmetry.

In triclinic symmetry, two mutually perpendicular raw decoupling planes can occur. Decoupling planes intersecting under an oblique angle are possible if the stiffnesses satisfy a number of constraints.

CONDICIONES PARA LA PRESENCIA DE PLANOS DESACOPLADOS EN MATERIALES ANISOTRÓPICOS ELÁSTICOS

Los planos de simetría son identificados a menudo por la existencia de una polarización pura de planos cruzados. Sin embargo, esta polarización puede ocurrir sin que el plano sea un plano de simetría. Los planos que soportan una polarización de planos cruzados son denominados "planos desacoplados", dado que el sistema de tres ecuaciones lineales acopladas en la dirección de los cosenos del vector de polarización, se desacopla en una ecuación única de plano cruzado y en un par acoplado de ecuaciones dentro del plano. Sólo en caso que la dirección perpendicular a un plano de desacoplamiento sea una "dirección longitudinal" (es decir, si en la dirección sólo hay ondas P y S puras) se trata de un plano de simetría. Sin la observación de la dirección longitudinal asociada, un plano de desacoplamiento "en bruto" puede ser erróneamente interpretado como plano de simetría.

El plano perpendicular a la dirección i es un plano de desacoplamiento si en la notación de cuatro subíndices todas las rigideces con un i único de subíndice han desaparecido; la dirección i es una dirección longitudinal si todas las rigideces con tres i de subíndice han desaparecido. En medios con simetría ortorrómbica o superior todas las rigideces con cualquier subíndice, único o triple, han desaparecido. Por lo tanto, los planos de desacoplamiento en bruto sólo pueden producirse en medios de simetría monoclínica o triclínica.

En simetría triclínica, pueden aparecer dos planos de desacoplamiento en bruto mutuamente perpendiculares. Pueden producirse planos de desacoplamiento con intersección en ángulo oblicuo si las rigideces satisfacen un cierto número de exigencias.

INTRODUCTION

Symmetry planes are an important tool for the determination of the symmetry of an elastic material and its orientation. The standard tool for the detection of symmetry planes is the inspection of polarization patterns observed either by surface seismics or in a VSP survey. Every symmetry plane is marked by a line-up of "cross-plane" and "in-plane" polarizations. The cross-plane polarization is often referred to as SH polarization, though in a strict sense this term should be applied only for vertical planes of symmetry.

Recently it was shown by Dellinger (informal message on the "anisotropists net-work") that pure cross polarization can exist for planes that are not planes of symmetry. This case was further informally discussed by Schoenberg. The short name "decoupling plane" is used since for restriction of the wave normal to such a plane the system of three coupled homogeneous linear equation in the direction cosines of the polarization vector (the Kelvin-Christoffel system) "decouples" into a separate linear equation for the cross-plane polarization and a system of two coupled linear equations for the two in-plane (or *qP*- and *qSV*-) polarizations (see section "definition"). Note that these are pure *P*- and *SV*-waves in "longitudinal" directions only. If there are pure *P*- and *SV*-waves in all di-rections in the plane, the plane is either the "main" symmetry plane in a transversely isotropic medium, or the medium is isotropic.

Since a "raw" decoupling plane could be mistakenly identified as a symmetry plane (and thus lead to erroneous interpretation), this paper discusses the conditions under which such planes can exist.

1 DEFINITIONS

A plane is a symmetry plane if—and only if—the following two conditions are satisfied:

– In all directions in the plane, purely transverse waves with cross polarization (and *qP* and *qS* waves with in-plane polarization) can propagate.

  The existence of pure cross-plane *S*-waves implies that for propagation in this plane the system of three coupled linear equations in the polarization directions—the Kelvin-Christoffel system—decouples into a single equation describing cross polarization and a coupled pair describing in-plane polarization

(Eqs. (1) and (2)). For this reason the planes can be called “decoupling planes”.

– The direction perpendicular to the plane is a longitudinal direction, i.e., a direction in which a purely longitudinal wave (and by implication two purely transverse waves) can propagate.

Without loss of generality we assume that the plane in question is a coordinate plane. Since no *a priori* assumption about the orientation of the coordinate system is made, one can always place the *j*- and *k*- direction into the plane. In this context we regard *i, j, k* as distinct fixed integers between 1 and 3, with no summation implied. Let α and β denote unit vectors in the direction of the displacement vector and the propagation vector, respectively, and let c_{ijkl} be the tensor of density-normalized stiffnesses. The Kelvin-Christoffel system for propagation restricted to the *jk*-plane is (since the component $\beta_i = 0$):

$$
\begin{aligned}
&(C_{ijij}\beta_j^2 + 2C_{ijik}\beta_j\beta_k + C_{ikik}\beta_k^2 - \nu^2)\,\alpha_i \\
&\quad + (C_{ijjj}\beta_j^2 + (C_{ikjj} + C_{ijjk})\beta_j\beta_k + C_{ikjk}\beta_k^2)\,\alpha_j \\
&\quad + (C_{ijjk}\beta_j^2 + (C_{ijkk} + C_{ikjk})\beta_j\beta_k + C_{ikkk}\beta_k^2)\,\alpha_k = 0 \\
&(C_{ijjj}\beta_j^2 + (C_{ikjj} + C_{ijjk})\beta_j\beta_k + C_{ikjk}\beta_k^2)\,\alpha_i \\
&\quad + (C_{jjjj}\beta_j^2 + 2C_{jjjk}\beta_j\beta_k + C_{jkjk}\beta_k^2 - \nu_2)\,\alpha_j \qquad (1)\\
&\quad + (C_{jjjk}\beta_j^2 + (C_{jjkk} + C_{jkjk})\beta_j\beta_k + C_{jkkk}\beta_k^2)\,\alpha_k = 0 \\
&(C_{ijjk}\beta_j^2 + (C_{ijkk} + C_{ikjk})\beta_j\beta_k + C_{ikkk}\beta_k^2)\,\alpha_i \\
&\quad + (C_{jjjk}\beta_j^2 + (C_{jjkk} + C_{jkjk})\beta_j\beta_k + C_{jkkk}\beta_k^2)\,\alpha_j \\
&\quad + (C_{jkjk}\beta_j^2 + 2C_{jkkk}\beta_j\beta_k + C_{kkkk}\beta_k^2 - \nu^2)\,\alpha_{ik} = 0
\end{aligned}
$$

System (1) decouples if all stiffnesses with a single subscript *i* vanish:

$$
\begin{aligned}
&(C_{ijij}\beta_j^2 + 2C_{ijik}\beta_j\beta_k + C_{ikik}\beta_k^2 - \nu^2)\,\alpha_i = 0 \\
&(C_{jjjj}\beta_j^2 + 2C_{jjjk}\beta_j\beta_k + C_{jkjk}\beta_k^2 - \nu_2)\,\alpha_j \\
&\quad + (C_{jjjk}\beta_j^2 + (C_{jjkk} + C_{jkjk})\beta_j\beta_k + C_{jkkk}\beta_k^2)\,\alpha_k = 0 \\
&(C_{jjjk}\beta_j^2 + (C_{jjkk} + C_{jkjk})\beta_j\beta_k + C_{jkkk}\beta_k^2)\,\alpha_j \qquad (2)\\
&\quad + (C_{jkjk}\beta_j^2 + 2C_{jkkk}\beta_j\beta_k + C_{kkkk}\beta_k^2 - \nu^2)\,\alpha_{ik} = 0
\end{aligned}
$$

The Kelvin-Christoffel system for propagation in the *i*-direction, i.e., the direction perpendicular to the *jk*-plane is (because of $\beta_i = 1$):

$$
\begin{aligned}
&(c_{iiii} - \nu^2)\,\alpha_i + c_{iiij}\,\alpha_j + c_{iiik}\,\alpha_k = 0 \\
&c_{iiij}\,\alpha_i + (c_{jjjj} - \nu^2)\,\alpha_j + c_{ijik}\,\alpha_k = 0 \qquad (3)\\
&c_{iiik}\,\alpha_i + c_{ijik}\,\alpha_j + (c_{ikik} - \nu^2)\,\alpha_{ik} = 0
\end{aligned}
$$

The *i*-direction is a longitudinal direction if all stiffnesses with three subscripts *i* vanish:

$$
\begin{aligned}
&(C_{iiii} - \nu^2)\,\alpha_i = 0 \\
&(C_{jjjj} - \nu^2)\,\alpha_j + C_{ijik}\,\alpha_k = 0 \qquad (4)\\
&C_{ijik}\,\alpha_j + (C_{ikik} - \nu^2)\,\alpha_{ik} = 0
\end{aligned}
$$

The two conditions above are obvious consequences of the geometric definition of the symmetry plane and thus sufficient conditions. That the conditions are also necessary is shown as follows: if all components of the stiffness tensor with an odd number (either 1 or 3) of subscripts *i* vanish, all non-zero components of the stiffness tensor have an even number of subscripts *i*. This makes the tensor invariant against a reversal of the direction of the *i*-axis, since for each occurrence of a subscript *i* the component is multiplied by $\cos \pi = -1$. An even number $2n$ ($n \in \{0, 1, 2\}$) of subscripts *i* implies multiplication with $(-1)^{2n} = 1$. On the other hand, any term with an odd number of subscripts *i* would be multiplied by –1, i.e., the symmetry would be broken.

The number and the arrangement of the symmetry planes uniquely determine the symmetry class of the medium and its orientation in space.

2 “RAW” DECOUPLING PLANES AND “FREE” LONGITUDINAL DIRECTIONS

If and only if both conditions are satisfied—i.e., if the plane is a decoupling plane and if the direction perpendicular to it is a longitudinal direction—the plane is a plane of symmetry. Simultaneous observation of both conditions, however, is not an easy matter, since the corresponding acquisition apertures are widely separated (Fig. 1). It is thus important to find out what observation of either condition without the other implies.

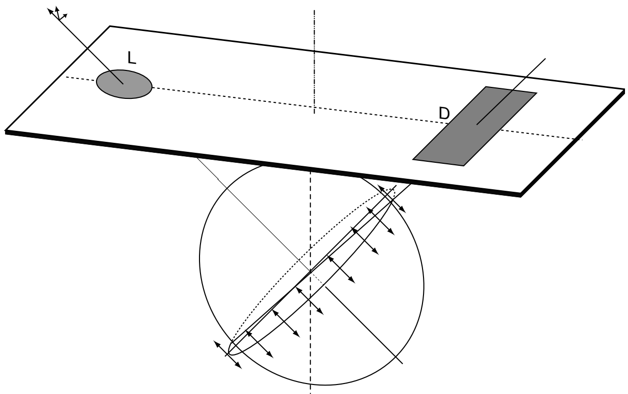


Figure 1

The acquisition apertures for the complete identification of a symmetry plane. Cross-plane polarization and the corresponding longitudinal direction have to be observed at two widely separated locations.

2.1 Longitudinal Directions

Kolodner (1966) has shown that every elastic medium has at least three longitudinal directions. Since there are media with less than three (or without any) planes of symmetry, it is obvious that longitudinal directions without the accompanying decoupling plane exist. Kolodner's argument goes at follows: in a longitudinal direction the wave normal and one of the polarization vectors are collinear. For such a direction the Kelvin-Christoffel equation (where now the summation convention is implied) takes the form:

$$c_{ijkl}\,\beta_l\beta_j\,\alpha_k - \nu^2\,\alpha_i = 0 \quad \underset{\text{longitudinal direction}}{\Rightarrow} \quad c_{ijkl}\,\beta_l\beta_j\,\beta_k - \nu^2\,\beta_i = 0 \tag{5}$$

Next, consider the "characteristic quartic" of the elastic tensor:

$$f(x_i) = c_{ijkl}\,x_i\,x_j\,x_k\,x_l = 1 \tag{6}$$

where $\boldsymbol{x}$ is a position vector of a closed surface, the "characteristic surface" representing (some of the aspects of) the elastic tensor c_{ijkl} (not all aspects can be represented, since a homogeneous quartic in three dimensions has only 15 distinct terms, while the general elastic tensor has more than 15 significant components). An example of a characteristic surface is shown in Figure 2.

The normal to the characteristic surface is:

$$\frac{\partial f(x_i)}{\partial x_q} = c_{qjkl}\,x_j\,x_k\,x_l \tag{7}$$

and the normal is collinear with the radius vector if:

$$c_{qjkl}\,x_j\,x_k\,x_l = \lambda\,x_i \tag{8}$$

In the directions satisfying (8) the radius vector is stationary. Comparison of (8) with final form of (5) shows that the longitudinal directions of the elastic medium are the "stationary directions" (maxima, minima and saddle points) of the characteristic surface. It can be shown that a smooth surface with central symmetry has at least three pairs of stationary directions (a pair consists of a direction and its opposite). In the current context an appeal to geometric intuition is sufficient: assume that the surface is not a surface of rotation, so that there are distinct largest and smallest diameters, indicating two stationary directions. Choose the plane spanned by these two diameters as a start and rotate it about the largest diameter. Monitor the length of the smallest diameter of the curve of intersection. Since in the starting plane the length is the *global* minimum, all these intermediate "smallest" lengths must be at least equal to it. The maximum of all intermediate smallest diameters indicates a direction where the derivative with respect to two independent parameters vanish, i.e., the third stationary direction. Thus any elastic medium must have at least three longitudinal directions. Details on this geometric argument can be found in Helbig (1993 and 1994).

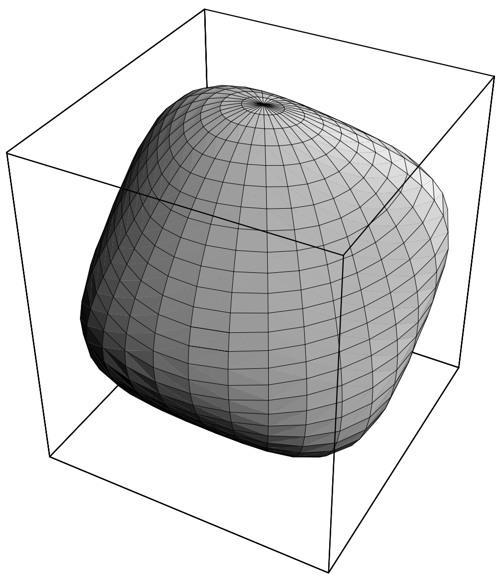


Figure 2

Characteristic quartic surface of a triclinic medium. Every place on this surface where the distance from the center is stationary (i.e., normal parallel to the radius vector) corresponds to a longitudinal direction in the elastic medium.

2.2 Decoupling Planes

The condition that all stiffnesses with one subscript *i* vanish applies to four-subscript notation. For the more common two-subscript notation this translates as follows:

23-plane ⊥ to 1-direction

$c_{25} = 0,\ c_{26} = 0,\ c_{35} = 0$

$c_{36} = 0,\ c_{45} = 0,\ c_{46} = 0$

13-plane ⊥ to 2-direction

$c_{14} = 0,\ c_{16} = 0,\ c_{34} = 0$

$c_{36} = 0,\ c_{45} = 0,\ c_{56} = 0$

12-plane ⊥ to 3-direction

$c_{14} = 0,\ c_{15} = 0,\ c_{24} = 0$

$c_{25} = 0,\ c_{46} = 0,\ c_{56} = 0$

The corresponding pattern of non-zero stiffnesses in the 6 × 6 Voigt-array (or the 6 × 6 Kelvin-tensor) is shown in Figure 3. Obviously, two coordinate planes may be co-existing decoupling planes in a triclinic medium. The same holds for monoclinic symmetry, as shown in Figure 4, where one of the two decoupling planes has been "elevated" to a plane of symmetry by adding a longitudinal direction perpendicular to it. Further below the conditions for multiple raw decoupling planes are discussed. This aspect is included for the sake of completeness. However, it might be possible to draw conclusions concerning symmetry and orientation from the relative attitude of such planes.

2.3 Decoupling Planes that Are Not Mutually Perpendicular

To investigate the possibility of two co-existing raw decoupling planes meeting at angles other than π/2, we assume the first plane to be the 13-plane and the 3-axis to be the line of intersection between the two planes. Let the two planes include and angle α. The 13-plane is a decoupling plane if $c_{14} = 0$, $c_{16} = 0$, $c_{34} = 0$, $c_{36} = 0$, $c_{45} = 0$ and $c_{56} = 0$. The second plane is a decoupling plane if also:

$$c'_{34} = c_{35} \sin\alpha = 0$$

$$c'_{36} = \frac{c_{13} - c_{23}}{2} \sin 2\alpha = 0$$

$$c'_{45} = \frac{c_{44} - c_{55}}{2} \sin 2\alpha = 0$$

$$\begin{aligned} c'_{14} = {} & \frac{c_{24}}{4}\cos\alpha + \frac{c_{24}}{4}\cos 3\alpha \\ & + \frac{-c_{15} - 3c_{25} + 2c_{46}}{4}\sin\alpha \\ & + \frac{-c_{15} + c_{25} + 2c_{46}}{4}\sin 3\alpha = 0 \\ c'_{56} = {} & \frac{c_{24}}{4}\cos\alpha + \frac{c_{24}}{4}\cos 3\alpha \\ & + \frac{-c_{15} + c_{25} - 2c_{46}}{4}\sin\alpha \\ & + \frac{-c_{15} + c_{25} + 2c_{46}}{4}\sin 3\alpha = 0 \end{aligned} \tag{9}$$

and:

$$c'_{16} = \frac{c_{26}}{2}\cos 2\alpha - \frac{c_{26}}{2}\cos 4\alpha - \frac{c_{11} - c_{22}}{2}\sin 2\alpha - \frac{d}{4}\sin 4\alpha = 0$$

with:

$$d = \frac{c_{11} - c_{22}}{2} - c_{12} - 2c_{66}$$

In (9), the Fourier-sequence form of the Bond relations has been used together with the vanishing of the six stiffnesses listed above.

Since α = π/2 and α = π are excluded, it follows immediately from the first three equations of (9) that:

$$c_{35} = 0,\ c_{13} = c_{23},\ c_{44} = c_{55} \tag{10}$$

Subtraction of c'_{14} and c'_{56} yields:

$$(-c_{25} + c_{46})\sin\alpha = 0 \tag{11}$$

from which one gets:

$$c_{25} = c_{46} \tag{12}$$

Summation of c'_{14} and c'_{56} yields:

$$\frac{c_{24}}{2}(\cos\alpha - \cos 3\alpha) - \frac{c_{15} - c_{25}}{2}\sin\alpha - \frac{c_{15} - 3c_{25}}{2}\sin 3\alpha = 0 \tag{13}$$

which reduces to:

$$\sin\alpha\,(-(c_{15} - c_{25}) - (c_{15} - 3c_{25})\cos 2\alpha + c_{24}\sin 2\alpha) = 0$$
$$\Rightarrow -(c_{15} - c_{25}) - (c_{15} - 3c_{25})\cos 2\alpha + c_{24}\sin 2\alpha = 0 \tag{14}$$

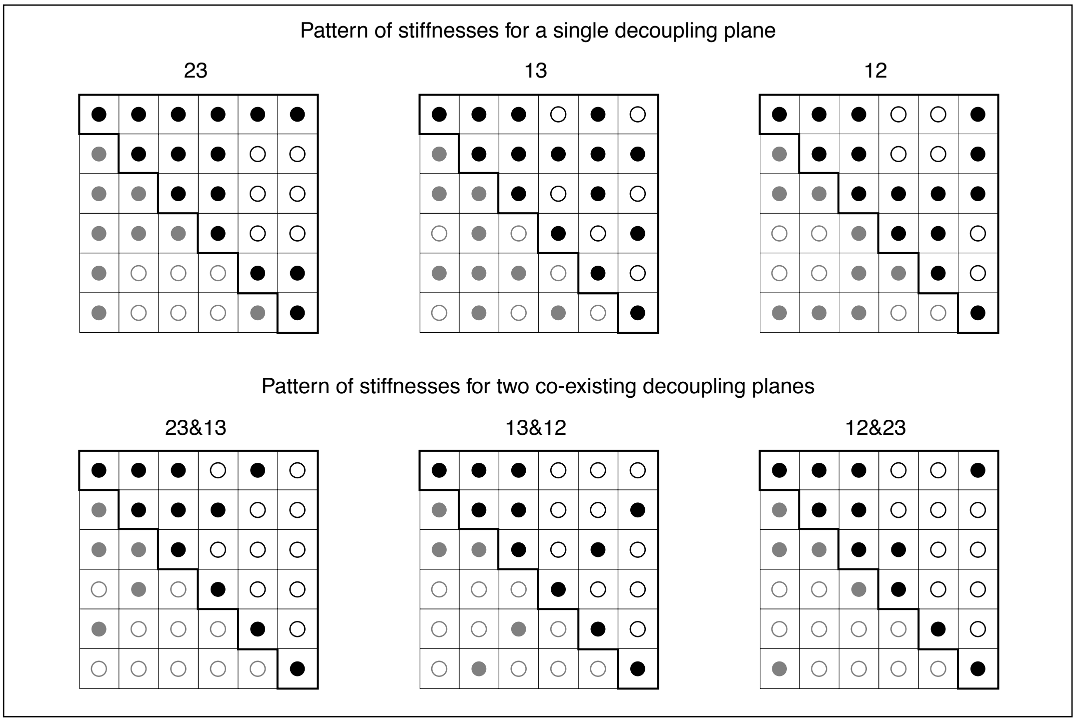
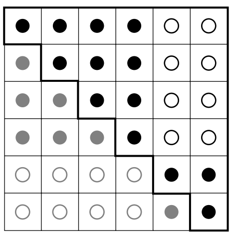


Figure 3

Pattern of stiffnesses for a single decoupling plane (top) and two perpendicular co-existing decoupling planes (bottom). Open points indicate stiffnesses that must vanish to make the plane a decoupling one. Filled points indicate non-zero stiffnesses. Grey filling indicates that the stiffness is identical to another entry.

- The addition of a longitudinal direction perpendicular to a decoupling plane makes the decoupling plane a plane of symmetry. E.g., for D(23)&L(1) one has the pattern:

D(23)&L(1)

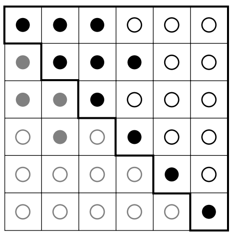


This is the pattern of a monoclinic medium with the two-fold axis in 1-direction (the 23-plane a plane of symmetry).

The symmetry follows also directly from the fact that all subscripts “1” occur in even multiplicity (0, 2, or 4). Thus a reversal of the 1-axis has no effect.

- The combination of a decoupling plane with a longitudinal direction parallel to it is also possible. Thus a monoclinic medium may have in addition to the symmetry plane a second decoupling plane:

D(23)&D(13)&L(1)



This stiffness matrix has still monoclinic symmetry. Two-fold axis in 1-direction, 13-plane is additional decoupling plane.

Figure 4

Pattern of stiffnesses for a single decoupling plane perpendicular to a single plane of symmetry.

The condition that $c'_{16} = 0$ leads to:

$$\frac{c_{26}}{2}(\cos 2\alpha - \cos 4\alpha) - \frac{c_{11} - c_{22}}{4}\sin 2\alpha - \frac{d}{4}\sin 4\alpha = 0 \qquad (15)$$

which reduces to:

$$c_{26}\sin\alpha\sin 3\alpha - \frac{c_{11} - c_{22}}{4}\sin 2\alpha - \frac{d}{4}\sin 4\alpha = 0 \qquad (16)$$

Equations (10), (12), (14) and (16) have to be satisfied for some oblique angle α for a second decoupling plane to exist. Not much more can be said beyond the fact that the medium must be monoclinic or triclinic for at least one of the two planes not to be a plane of symmetry.

One special case deserves some further attention (though it has no relevance in an exploration context): a trigonal medium is characterized by three planes of symmetry intersecting in a common line and making angles of $\pi/3$. Is it possible for raw decoupling planes to intersect a common line at an angle of $\pi/3$? If so, one could distinguish two cases: if neither plane is a plane of symmetry, the system could exist of two such planes. If one is a plane of symmetry, the other plane is "mirrored" so that the system consists of three planes intersecting an angles of $\pi/3$, two of them raw decoupling planes. If both planes intersecting at $\pi/3$ are planes of symmetry, there is a third plane of symmetry and thus the medium is trigonal.

For $\alpha = \pi/3$ Equations (14) and (16) take the form:

$$c_{15} + c_{25} = \sqrt{3}\, c_{24} \qquad (17)$$

and:

$$c_{11} - c_{22} = \frac{c_{11} + c_{22}}{2} - c_{12} - 2c_{66} \qquad (18)$$

CONCLUSIONS

Raw decoupling planes—planes that support cross-plane polarization without being planes of symmetry—can only exist in monoclinic and triclinic media. Without the corresponding observation of a longitudinal direction perpendicular to the plane, the observation of persistent cross-plane polarization may indicate a decoupling plane rather than a plane of symmetry. Since this can happen only for low symmetry, such an occurrence in an exploration context is unlikely, but one still has to guard against a erroneous interpretation.

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