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Natural Vibration Analysis of Tensioned Risers by Segmentation Method

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Résumé — Analyse des vibrations propres des risers tendus par la méthode de segmentation — Dans le but de l'évaluation des vibrations des risers induites par vortex (VIV), les vibrations modales d'un riser uniforme tendu en eau profonde sont considérées. Du fait de la faible rigidité des risers, les vibrations du type câble et poutre sont analysées. Les solutions exactes et asymptotiques de l'équation différentielle de câble sont présentées. Les vibrations de la poutre sont déterminées en modifiant la solution câble pour chaque demi-longueur d'onde du mode propre. La procédure analytique simplifiée est vérifiée par la méthode des éléments finis. Les conclusions utiles sur le comportement dynamique des risers dans le domaine des hautes et basses fréquences sont présentées.

***Abstract — Natural Vibration Analysis of Tensioned Risers by Segmentation Method** — The natural vibration of deepwater tensioned uniform risers is analysed. The riser is considered as both a cable and a beam due to the fairly weak flexural stiffness. The exact and asymptotic solutions of the cable differential equation are presented. The beam vibration is determined by modifying the cable solution for each half-wave of the natural mode which are called* segments. *The simplified analytical procedure is verified by the finite element method. Very valuable conclusions are drawn on riser dynamic behaviour in the lower and higher frequency domain. The main application of the study is the assessment of forced vortex-induced riser vibration.*

NOMENCLATURE

A, B integration constants
C coefficient of mode envelope
E Young's modulus
I moment of inertia of cross-section
J_0 Bessel function of the first kind and order zero
L riser length
M bending moment
Q_k hypothetical tension force due to bending of the k-th segment
T_b bottom tension force
T_e equivalent tension force
T_{ek} equivalent tension force of the k-th segment
T_k nodal tension force
T_t top tension force
T_x tension force at height x
X amplitude function
Y modal amplitude
Y_0 Bessel functions of the second kind and order zero
a parameter
i iteration step
j ordinary anti-node number
k ordinary number of mode half-wave (segment), ordinary node number
l_k segment length
m virtual mass (riser mass + added mass) per unit length
n mode number
p, q variable coefficients
t time
w riser weight per unit length
x vertical coordinate
y vibration displacement
z argument of Bessel functions
α argument
β coefficient
γ phase angle
ε small quantity, error
ζ argument
κ curvature
φ rotation angle
ω_n natural frequency of the n-th mode
$(...)^*$ modified quantity
$(...)'$ beam quantity

INTRODUCTION

Risers are one of the basic elements of offshore installations, which are used for drilling, production and intervention. They are exposed to the strong influences of environmental conditions in unshielded deep waters. Risers are very sensitive to external excitation due to their slenderness (Patel *et al.*, 1995; Park *et al.*, 2002; Hong and Koterayama, 2003; Chatjigeorgiou *et al.*, 2003).

The main problem in design and operation of deepwater risers nowadays is vortex-induced vibration (VIV) in a severe current. According to full scale measurements, the vortex-shedding frequency is adopted to the one of natural frequencies of the riser. This phenomenon is known as *lock-in*, which means that the vortex shedding frequency is locked into the riser's natural frequency. The amplitude of such resonant elastic response is of the order of magnitude of the riser diameter. This excessive vibration may cause serious fatigue damage, especially in the higher frequency domain (Bowman and Howells, 1998; Howells and Lim, 1998; Khalak and Williamson 1999; Goverdhan and Williamson, 2000; Bell *et al.*, 2002; Yamamoto *et al.*, 2004.).

The theoretical consideration shows that the envelope of riser natural modes is damped from the sea bottom to the riser top. On the other hand, the vortex energy distribution is reduced from the sea surface to the bottom. Thus, the envelope of the forced vibration amplitude is almost constant per riser height.

Besides the stream-wise motion, the cross-stream vibration is also excited due to alternate vortex generation at the cylinder's sides. Thus, the vortex-induced riser vibration is analysed as a single or two degrees of freedom system with natural modes. Time solution is achieved by the time integration of the rather complex governing equations of motion. Therefore, for the vortex-induced forced vibration a reliable analysis of the riser natural vibration is very important.

Since a deepwater riser is a slender structure, the influence of flexural stiffness on the few lowest natural modes is very low and rises for higher modes. Therefore, in order to simplify the procedure the riser vibration analysis is usually based on cable vibration theory and its analogical extension to the beam vibration problems. Thus, a rather simplified analytical approach is described in Sparks (2002) in order to explain the physics of riser dynamics. However, it is possible to solve the problem by the same approach but in a more sophisticated way. This is done in this paper, where only minor simplifications are introduced. The influence of vibration parameters based on their relations in governing equations is analysed in detail and some additional explanations of riser dynamic behaviour are given.

The procedure is illustrated in a case study of a uniform deepwater riser with simply supported edges. The riser is considered as a cable and beam with zero and small flexural stiffness respectively. The differential equation of cable vibration is solved exactly by Bessel's functions and approximately by the asymptotic approach.

In order to analyse the beam vibration, the cable solution is modified following an idea elaborated in Sparks (2002). However, the procedure is formulated in a somewhat different way. The concept of an equivalent tension force is introduced as substitution of linearly varying real force. The obtained results by the simplified procedure called *the segmentation method* are validated by the finite element solution.

1 CABLE VIBRATION

1.1 Exact Solution

The differential equation of natural transverse vibration of a tensioned cable with neglecting flexural stiffness yields (Timoshenko, 1955; Clough and Penzien, 1975):

$$T_x \frac{\partial^2 y}{\partial x^2} + w \frac{\partial y}{\partial x} - m \frac{\partial^2 y}{\partial t^2} = 0 \tag{1}$$

where T_x is tension force, w is distributed weight, m is specific mass, y is displacement, x is vertical coordinate and t is time. The tension force consists of a constant and varying part (Spark, 2002):

$$T_x = T_b + wx = T_b + (T_t - T_b)\frac{x}{L} \tag{2}$$

where T_b and T_t are the bottom and the top value respectively. Natural vibration is of a harmonic nature:

$$\gamma = Y \sin \omega_n t \tag{3}$$

where Y is the mode function and ω_n is the natural frequency. By substituting (3) into (1) the modal equation is obtained:

$$T_x \frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + w \frac{\mathrm{d}Y}{\mathrm{d}x} + m\omega_n^2 Y = 0 \tag{4}$$

By following Sparks (2002), (4) may be written in the form:

$$a\left(x + \frac{T_b}{w}\right)\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + a\frac{\mathrm{d}Y}{\mathrm{d}x} + Y = 0 \tag{5}$$

where:

$$a = \frac{w}{m\omega_n^2} \tag{6}$$

Substitution:

$$x = \frac{az^2}{4} - \frac{T_b}{w} \tag{7}$$

transforms (5) into the standard form of Bessel's equation:

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}Y}{\mathrm{d}z} + Y = 0 \tag{8}$$

where according to (2), (6) and (7):

$$z = \frac{2\sqrt{mT_x}}{w}\omega_n \tag{9}$$

The solution of (8) yields:

$$Y = A J_0(z) + B Y_0(z) \tag{10}$$

where $J_0(z)$ and $Y_0(z)$ are Bessel functions of the first and second kind of order zero (Kreyszig, 1993; Abramowitz and Stegun, 1968). The relative value of the integration constants A and B, and natural frequency ω_n contained in the argument of Bessel functions (9) depend on the given boundary conditions.

1.2 Asymptotic Solution

In the case that the minimum value of argument z is large enough, what depends on the value of the bottom tension force T_b in (9), differential equation (8) may be solved approximately. By substituting:

$$Y = \frac{1}{\sqrt{z}} X \tag{11}$$

into (8) yields:

$$\frac{\mathrm{d}^2 X}{\mathrm{d}z^2} + \left(1 + \frac{1}{4z^2}\right) X = 0 \tag{12}$$

If the variable term $1/4z^2$ in (12) is small with respect to unity it may be replaced with its average value 2ε in the cable domain:

$$2\varepsilon = \frac{1}{z_t - z_b}\int_{z_b}^{z_t} \frac{\mathrm{d}z}{4z^2} = \frac{1}{4z_t z_b} \tag{13}$$

Thus, Equation (12) is transformed into a differential equation with constant coefficients:

$$\frac{\mathrm{d}^2 X}{\mathrm{d}z^2} + (1 + 2\varepsilon) X = 0 \tag{14}$$

and its solution reads:

$$X = A \sin\beta z + B \cos\beta z \tag{15}$$

where:

$$\beta = \sqrt{1 + 2\varepsilon} \approx 1 + \varepsilon \tag{16}$$

If the value of ε is very small, then it may be neglected and the asymptotic solution of the differential equation (8) according to (11) and (15) takes the form:

$$Y = \frac{1}{\sqrt{z}}(A \sin z + B \cos z) \tag{17}$$

The boundary conditions for the simply supported cable read $Y = 0$ at $z = z_t$ and $z = z_b$. This leads to the frequency equation:

$$\frac{1}{\sqrt{z_t z_b}} \sin(z_t - z_b) = 0 \tag{18}$$

and its eigenvalues:

$$z_t - z_b = n\pi \tag{19}$$

By employing (9) and (19) one finds the formula for the natural frequency:

$$\omega_n = \frac{n\pi w}{2\sqrt{m}\left[\sqrt{T_t} - \sqrt{T_b}\right]} = \frac{n\pi}{2L\sqrt{m}}\left[\sqrt{T_t} + \sqrt{T_b}\right] \tag{20}$$

After determining the relative values of the integration constants A and B based on the boundary conditions, the expression for the vibration mode (17) may be presented in the normalized form:

$$Y = \sqrt{\frac{z_b}{z}} \sin(z - z_b) \tag{21}$$

By employing (9), the mode equation (21) takes the form:

$$Y = \sqrt[4]{\frac{T_b}{T_x}} \sin(z - z_b) \tag{22}$$

It is interesting that the mode envelope does not depend on the mode number n, *i.e.* it is presented by the same function for all the natural modes. On the other hand, the argument z is the mode-dependent function (9) which, taking into account (20) reads:

$$z = \frac{\sqrt{T_x}}{\sqrt{T_t} - \sqrt{T_b}} n\pi \tag{23}$$

If the above determination of the natural frequencies is performed by employing more accurate solution of the differential equation, represented by formulae (11) and (15), then the corrected natural frequency reads:

$$\tilde{\omega}_n = \frac{\omega_n}{\beta} \approx (1 - \varepsilon)\omega_n \tag{24}$$

1.3 Coordinates of Vibration Nodes and Anti-Nodes

The n-th natural mode is shown in Figure 1, as well as the numbering of vibration nodes. Since at the k-th node the deflection $Y_k = 0$, according to (21) it must be:

$$z_k - z_b = (k-1)\pi, \qquad k = 1, 2 \ldots n+1 \tag{25}$$

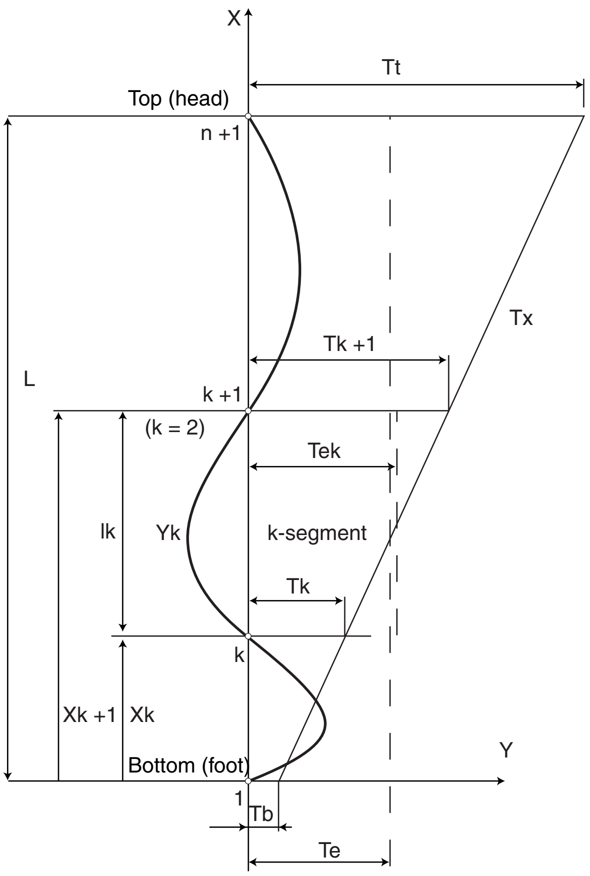

Figure 1

Cable vibration mode.

By employing (2) and (23) one obtains for the k-th node the coordinate:

$$\frac{x_k}{L} = \frac{\left[\frac{k-1}{n}\left(\sqrt{T_t} - \sqrt{T_b}\right) + \sqrt{T_b}\right]^2 - T_b}{T_t - T_b}, \quad k = 1, 2 \ldots n+1 \tag{26}$$

The length of the k-th half-wave of n-th mode is:

$$l_k = x_{k+1} - x_k, \qquad k = 1, 2 \ldots n \tag{27}$$

Anti-nodes are points of maximum deflection and their coordinates are determined from condition $\mathrm{d}Y / \mathrm{d}x = 0$. Referring to (21) yields:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \sqrt{\frac{z_b}{z}}\left[\cos(z - z_b) - \frac{1}{2z}\sin(z - z_b)\right]\frac{\mathrm{d}z}{\mathrm{d}x} \tag{28}$$

The application of the fundamental identities of trigonometric functions of two arguments (Bronshtejn and Semendjajev, 1975) transforms the condition $\mathrm{d}Y / \mathrm{d}x = 0$ into:

$$\cos(\gamma + z - z_b) = 0 \tag{29}$$

where:

$$\gamma = \mathrm{arctg}\frac{1}{2z} \tag{30}$$

Since γ is very small it may be neglected and (29) is approximately satisfied if:

$$z - z_b = \left(j - \frac{1}{2}\right)\pi \qquad j = 1, 2 \dots n \tag{31}$$

Thus, the coordinate of the j-th anti-node similarly to (26) takes an approximate value:

$$\frac{x_j}{L} \approx \frac{\left[\frac{j-\frac{1}{2}}{n}\left(\sqrt{T_t} - \sqrt{T_b}\right) + \sqrt{T_b}\right]^2 - T_b}{T_t - T_b} \qquad j = 1, 2 \dots n \tag{32}$$

1.4 Rotation Angle and Curvature

Rotation angle is presented by (28) and (29), *i.e.*

$$\varphi = \frac{\mathrm{d}Y}{\mathrm{d}x} = \sqrt{\frac{z_b}{z}}\cos(\gamma + z - z_b)\frac{\mathrm{d}z}{\mathrm{d}x} \tag{33}$$

where according to (2) and (23):

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{n\pi}{2L}\frac{\sqrt{T_t} + \sqrt{T_b}}{\sqrt{T_x}} \tag{34}$$

Maximum values of φ occur in the vibration nodes:

$$\varphi_k = \sqrt{\frac{z_b}{z_k}}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_k \tag{35}$$

According to definition and (28) the cable curvature yields:

$$\kappa = \frac{\mathrm{d}^2Y}{\mathrm{d}x^2} = -\sqrt{\frac{z_b}{z}}\left\{\left[\left(1 - \frac{3}{4z^2}\right)\sin(z - z_b) + \frac{1}{z}\cos(z - z_b)\right]\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + \right.$$
$$\left. + \left[\frac{1}{2z}\sin(z - z_b) - \cos(z - z_b)\right]\frac{\mathrm{d}^2z}{\mathrm{d}x^2}\right\} \tag{36}$$

The curvature is maximum in anti-nodes, xj Eq. (32), where $\sin(z_j - z_b) = 1$ and $\cos(z_j - z_b) = 0$. Thus, one finds:

$$\kappa_j = -\sqrt{\frac{z_b}{z_j}}\left[\left(1 - \frac{3}{4z_j^2}\right)\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_j^2 + \frac{1}{2z_j}\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}\right)_j\right] \tag{37}$$

where dz/dx is represented by (34), while:

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = -\frac{n\pi}{4L^2}\frac{\left(\sqrt{T_t} + \sqrt{T_b}\right)(T_t - T_b)}{T_x^{3/2}} \tag{38}$$

1.5 Definition of Equivalent Tension Force

Instead of variable tension force T_x and specific weight w let us express the differential equation of cable vibration (4) with equivalent constant force T_e for simplicity reason, which gives the same natural frequencies. In that case:

$$T_e\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} + m\omega_n^2Y = 0 \tag{39}$$

and natural frequency for the simply supported ends takes the value:

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T_e}{m}} \tag{40}$$

The natural frequency for the case with a variable tension force is represented by (20). A formula for the equivalent tension force is determined by equalising (20) and (40) that leads to:

$$T_e = \frac{(Lw)^2}{4\left(\sqrt{T_t} - \sqrt{T_b}\right)^2} = \frac{1}{4}\left(\sqrt{T_t} + \sqrt{T_b}\right)^2 \tag{41}$$

1.6 Mode Half-Wave Characteristics

Since each k-th half-wave of a vibration mode has the same natural frequency, the formula (41) is generally valid for all half-waves, Figure 1, *i.e.*:

$$T_{e_k} = \frac{(l_k w)^2}{4\left(\sqrt{T_{k+1}} - \sqrt{T_k}\right)^2} = \frac{1}{4}\left(\sqrt{T_{k+1}} + \sqrt{T_k}\right)^2 \tag{42}$$

The equivalent forces are related to the vibration anti-nodes (Sparks, 2002). According to (19) for the mode adjacent nodes yields:

$$z_{k+1} - z_k = \pi \tag{43}$$

By substituting (23) into (42) one finds:

$$\sqrt{T_{k+1}} - \sqrt{T_k} = \frac{\sqrt{T_t} - \sqrt{T_b}}{n} \tag{44}$$

Furthermore, by inserting (44) into (42) and employing (41), a direct relation between half-wave equivalent force and its length is obtained:

$$\sqrt{T_{e_k}} = \frac{n w l_k}{2\left[\sqrt{T_t} - \sqrt{T_b}\right]} = \frac{n\sqrt{T_e}}{L}l_k \tag{45}$$

Finally, by summing up (45) for all the mode half-waves, and taking into account that $\sum_{k=1}^{n} l_k = L$, the condition for the cable equivalent forces yields:

$$\frac{1}{n}\sum_{k=1}^{n}\sqrt{T_{e_k}} = \sqrt{T_e} \tag{46}$$

Thus, the average value of the square root half-wave equivalent forces is equal to the square root of the cable equivalent force.

2 BEAM VIBRATION

2.1 Differential Equation

Differential equation of beam natural vibration, taking into account flexural stiffness, according to Timoshenko (1955) and Clough and Penzien (1975) yields:

$$EI\frac{\partial^4 y}{\partial x^4} - T_x\frac{\partial^2 y}{\partial x^2} - w\frac{\partial y}{\partial x} + m\frac{\partial^2 y}{\partial t^2} = 0 \tag{47}$$

Due to harmonic nature of natural vibration:

$$y = Y\sin\omega_n' t \tag{48}$$

and the modal differential equation takes the form:

$$EI\frac{\mathrm{d}^4 Y}{\mathrm{d}x^4} - T_x\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} - w\frac{\mathrm{d}Y}{\mathrm{d}x} - m\omega_n'^2 Y = 0 \tag{49}$$

The beam equation (49) is of the fourth order with a variable coefficient. The intension is to transform it into a second order equation and solve it by analogy with the cable solution. For that purpose let us express the term of the fourth derivation by a term of the second derivation:

$$EI\frac{\mathrm{d}^4 Y}{\mathrm{d}x^4} = -Q_x\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} \tag{50}$$

where Q_x is an unknown hypothetic tension force.

By substituting (50) into (49) yields:

$$T_x'\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + w\frac{\mathrm{d}Y}{\mathrm{d}x} + m\omega_n'^2 Y = 0 \tag{51}$$

where:

$$T_x' = T_x + Q_x \tag{52}$$

In addition (50) may be expressed by the bending moment:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + \frac{Q_x}{EI}M = 0 \tag{53}$$

Following the way of transformation of the cable equation in Section 1.1 let us introduce substitutions:

$$a' = \frac{w}{m\omega_n'^2} \tag{54}$$

$$x = \frac{a' z^{*2}}{4} - \frac{T_b}{w} \tag{55}$$

Thus, Equation (51) is transformed into the form:

$$\left(1 + \frac{4Q_x}{a' w z^{*2}}\right)\frac{\mathrm{d}^2 Y}{\mathrm{d}z^{*2}} + \left(1 - \frac{4Q_x}{a' w z^{*2}}\right)\frac{1}{z^*}\frac{\mathrm{d}Y}{\mathrm{d}z^*} + Y = 0 \tag{56}$$

where according to (2) and (55):

$$z^* = \frac{2\sqrt{mT_x}}{w}\omega_n' \tag{57}$$

Now, a new argument may be introduced:

$$z' = \frac{2\sqrt{mT_x'}}{w}\omega_n' \tag{58}$$

where T'_x is defined by (52). Equation (56) takes the form:

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}z'^2} + p(z^*, z')\frac{\mathrm{d}Y}{\mathrm{d}z'} + q(z^*, z')Y = 0 \tag{59}$$

where coefficients $p(z^*,z')$ and $q(z^*,z')$ are very complex implicit functions. They include the unknown force Q_x which should be determined from (53).

Due to the above reasons it is not possible to transform (59) into the form of a cable Equation (8) and apply analogical parameters and solution. Thus, a suggestion concerning this matter, given in Sparks (2002), leads to a rather rough approximate solution especially for higher modes. Therefore, the problem is solved in a more reliable way by the so-called *segmentation method*, following the idea presented in Sparks, 2002.

2.2 Segmentation Solution

In order to simplify the terminology the mode half-waves are called *segments* for short. Equation (51) with variable tension force T'_x may be expressed for the k-th segment by its constant equivalent force T'_{ek}, which ensures the same natural frequency:

$$T_{e_k}'\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} + m\omega_n'^2 Y = 0 \tag{60}$$

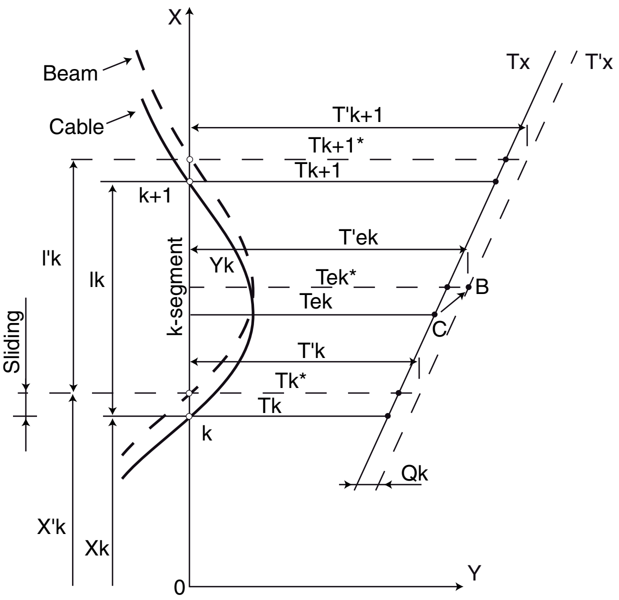

Figure 2

Cable and beam parameters.

where:

$$T'_{e_k} = T^*_{e_k} + Q_k \tag{61}$$

T'_{ek} is the equivalent segment force due to sliding of the nodes and anti-nodes, and Q_k is a hypothetic constant force as a result of flexural stiffness effects, Figure 2.

On the other hand, Equation (50) for the k-th segment, with its constant force Q_k, after double integration takes the form:

$$EI\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} + Q_kY = 0 \tag{62}$$

Differential equations (60) and (62) are of the same type with the common mode function Y. Therefore, their solutions have to be identical and that is achieved if their coefficients are proportional. This condition leads to the following formula for the definition of the unknown force Q_k:

$$Q_k = EI\,m\frac{\omega_n'^2}{T'_{e_k}} \tag{63}$$

The segment natural frequency according to (40) yields:

$$\omega'_n = \frac{\pi\sqrt{T'_{e_k}}}{l'_k\sqrt{m}} \tag{64}$$

The beam segment length from (64) is:

$$l'_k = \frac{\pi\sqrt{T'_{e_k}}}{\omega'_n\sqrt{m}} \tag{65}$$

Since the sum of the segment length has to be equal to the beam length, *i.e.* $\sum_{k=1}^{n} l'_k = L$, and since natural frequency ω'_n is common for all the segments, summing up (65) results in:

$$\omega'_n = \frac{\pi}{L\sqrt{m}}\sum_{k=1}^{n}\sqrt{T'_{e_k}} \tag{66}$$

The natural frequency of cable vibration ω_n Equation (40) may be incorporated into (66). In that way (66) takes the following form:

$$\omega'_n = \frac{\sum_{k=1}^{n}\sqrt{T'_{e_k}}}{n\sqrt{T_e}}\omega_n \tag{67}$$

Furthermore, by substituting (66) into (65) the segment length yields a more convenient form:

$$l'_k = \frac{L\sqrt{T'_{e_k}}}{\sum_{k=1}^{n}\sqrt{T'_{e_k}}} \tag{68}$$

Finally, by substituting (64) into (63) a simple formula for Q_k is obtained:

$$Q_k = \left(\frac{\pi}{l'_k}\right)^2 EI \tag{69}$$

All the beam vibration parameters are non-linearly mutually dependent, and therefore the problem has to be solved by an iteration procedure as elaborated in Appendix.

The beam mode shape is defined separately for each segment as solution of Equation (60), *i.e.*:

$$Y_k = \sin\left[\alpha_k(x - x'_k)\right] \tag{70}$$

where:

$$\alpha_k = \sqrt{\frac{m}{T'_{e_k}}}\omega'_n \tag{71}$$

and x'_k is the node coordinate. Formula (70) may be extended to hold the complete beam mode shape including the mode envelope in analogy with the cable solution (21):

$$Y = \sqrt{\frac{\zeta_b}{\zeta}}\sin(\zeta - \zeta_b) \tag{72}$$

where:

$$\zeta - \zeta_b = \alpha_k(x - x'_k) + \sum_{j=1}^{k-1}\alpha_j(x'_{j+1} - x'_j) \tag{73}$$

Since according to (64) and (71):

$$\alpha_k(x'_{k+1} - x'_k) = \alpha_k\,l'_k = \pi \tag{74}$$

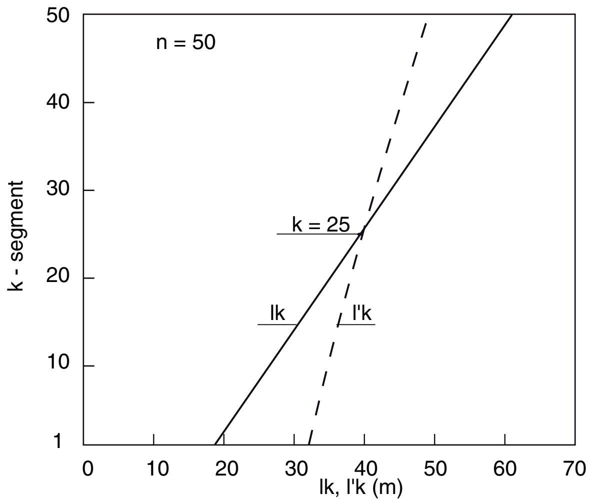

Figure 3

Cable and beam segment lengths.

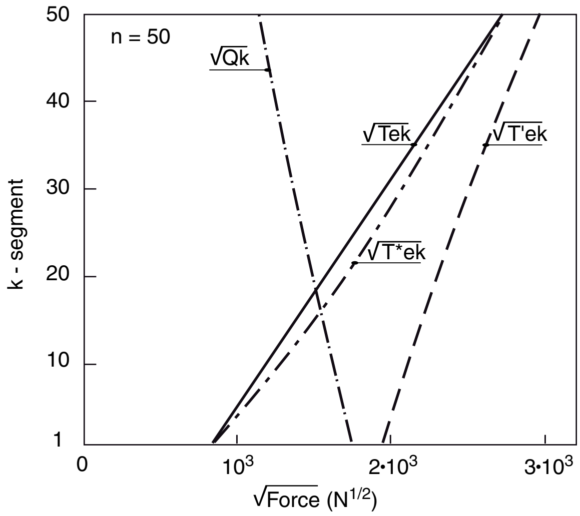

Figure 4

Cable and beam segment forces.

it follows from (73):

$$\zeta - \zeta_b = \alpha_k (x - x'_k) + (k-1)\pi \tag{75}$$

The difference between the coordinate boundary values yields:

$$\zeta_t - \zeta_b = n\pi \tag{76}$$

The unknown bottom value ζ_b may be determined empirically if the top value of the normalised mode envelope, C_t, is known from another source. Thus, according to (72):

$$\sqrt{\frac{\zeta_b}{\zeta_t}} = C_t \tag{77}$$

and by employing (76) one obtains:

$$\zeta_b = \frac{n\pi C_t^2}{1 - C_t^2} \tag{78}$$

The extended formula for the beam's natural mode (72) represents the solution of a hypothetic beam differential equation which is analogical to the cable equation (8).

3 ILLUSTRATIVE EXAMPLE

The application of the previously presented cable and beam vibration theory is illustrated in the case of a drilling riser analysed in Sparks (2002). Its main characteristics are the following:

Riser length	$L = 2000$ m
Flexural stiffness	$EI = 318.6 \cdot 10^6$ Nm^2
Linear weight	$w = 3433.5$ N/m
Total weight	$W = 6.867 \cdot 10^6$ N
Top tension	$T_t = 7.5537 \cdot 10^6$ N
Bottom tension	$T_b = 0.6867 \cdot 10^6$ N
Virtual mass (riser mass + added mass)	$m = 1200$ kg/m

In order to validate the obtained results by the asymptotic integration and segmentation method respectively, the cable vibration is also analysed by the exact method expressed by the Bessel functions, while beam vibration is performed by the finite element method (Zienkiewicz, 1971; Bathe, 1996). The values of natural frequencies are compared in Table 1.

TABLE 1
Riser natural frequencies, ω_n [s^{-1}]

	Cable ($EI = 0$)		Beam ($EI > 0$)	
Mode n	Exact Section 1.1	Asymptotic Section 1.2	Segmentation Section 2.2	FEM
1	0.07973	0.08108	0.08115	0.07983
2	0.16140	0.16217	0.16238	0.16176
3	0.24273	0.24326	0.24399	0.24370
4	0.32395	0.32435	0.32610	0.32602
5	0.40511	0.40544	0.40886	0.40891
10	0.81072	0.81089	0.83553	0.83580
20	1.62170	1.62179	1.77296	1.77331
30	2.43263	2.43268	2.84559	2.84630
40	3.24353	3.24357	4.07467	4.07600
50	4.05443	4.05447	5.47942	5.48210

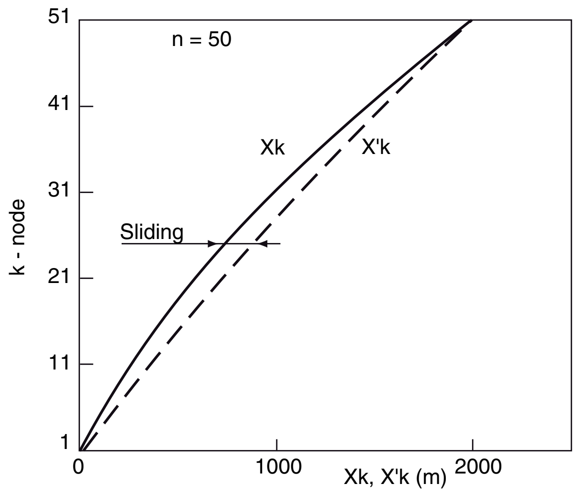

Figure 5

Cable and beam node coordinates.

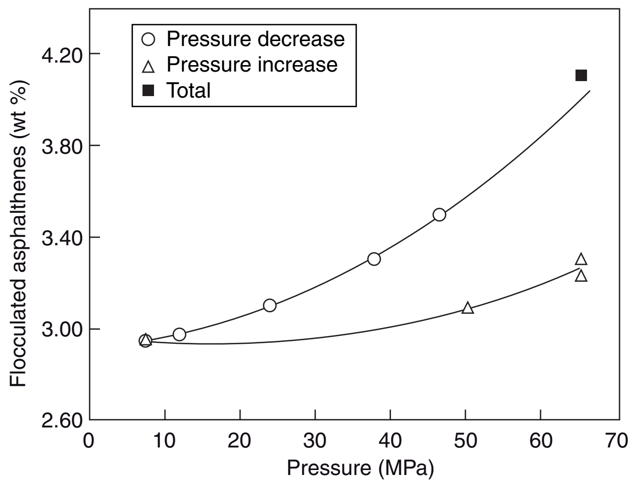

Figure 6

Dimensionless coordinates of cable and beam vibration nodes.

The asymptotic solution of cable vibration shows some discrepancies with respect to the exact solution, which are rapidly decreasing for higher modes. The estimated error of the first natural frequency reads 1.69%, which is fairly close to the predicted error ε = 2.05% according to Equation (13).

The beam natural frequencies are determined by the iterative segmentation method for the given accuracy of 10^{-7}. The iterated parameters converge uniformly. The number of iteration steps depends on the mode number. For instance, the given accuracy is achieved in 7 and 39 steps for $n = 10$ and 50 respectively.

The beam vibration analysis is also performed by the finite element method employing a special program developed within a Master's Thesis (Parunov, 1996). The FEM model consists of 200 beam elements of equal 10 m length that is lower than the minimum segment length. The value of the first natural frequency is very close to the exactly determined cable frequency, since influence of the flexural stiffness on the first natural mode is negligible.

In the lower frequency domain the beam natural frequencies determined by the segmentation method are somewhat higher than the FEM values. In the higher frequency domain the situation is opposite. Generally, the agreement between the results is very good, since for instance discrepancy between the 50-th natural frequency values is only 0.05%.

The numerical procedure of the segmentation method is illustrated in Table 2 in the case of the 50-th natural mode. The obtained results are also shown in Figures 3 and 4. The cable segment length, l_k, linearly increases from the bottom to the top, as well as the square root of the equivalent tension force, $\sqrt{T_{e_k}}$. The beam segment length, l'_k, is increased at the bottom and decreased at the top compared to the cable ones, while the length of the middle segment is almost unchanged, Figure 3. Due to the sliding of the beam nodes, Figure 2, the value of $\sqrt{T^*_{e_k}}$ is increased with respect to the cable values $\sqrt{T_{e_k}}$, Figure 4. The boundary increases are very small since sliding of the bottom and top segment is negligible. The hypothetical tension force due to flexural rigidity, Q_k, is of the same order of magnitude as the equivalent forces, Figure 4. It takes the maximum value at the bottom. As a result the increase of $\sqrt{T'_{e_k}}$ with respect to $\sqrt{T_{e_k}}$ is very large, much larger at the bottom than at the top.

TABLE 2

Riser parameters, Mode 50

Cable ($EI = 0$): $\omega_{50} = 4.05447\ s^{-1}$ Eq. (20)			Beam ($EI > 0$): $\omega'_{50} = 5.47942\ s^{-1}$ Eq. (66)			
Segment k	l_k [m] Eq. (27)	T_{ek} [N] Eq. (42)	Q_k [N] Eq. (69)	$\sqrt{T'_{e_k}}$ [N]	T'_{ek} [N] Eq. (61)	l_k' [m] Eq. (68)
50	61.04	$7.449 \cdot 10^6$	$1.308 \cdot 10^6$	$2.963 \cdot 10^3$	$8.777 \cdot 10^6$	49.03
40	52.45	$5.500 \cdot 10^6$	$1.550 \cdot 10^6$	$2.721 \cdot 10^3$	$7.405 \cdot 10^6$	45.04
30	43.86	$3.847 \cdot 10^6$	$1.846 \cdot 10^6$	$2.494 \cdot 10^3$	$6.220 \cdot 10^6$	41.28
20	35.28	$2.488 \cdot 10^6$	$2.200 \cdot 10^6$	$2.284 \cdot 10^3$	$5.217 \cdot 10^6$	37.80
10	26.69	$1.424 \cdot 10^6$	$2.615 \cdot 10^6$	$2.095 \cdot 10^3$	$4.389 \cdot 10^6$	34.67
1	18.96	$7.189 \cdot 10^5$	$3.037 \cdot 10^6$	$1.944 \cdot 10^3$	$3.779 \cdot 10^6$	32.18
Σ = 2000			Σ = $14.501 \cdot 10^3$			Σ = 2000

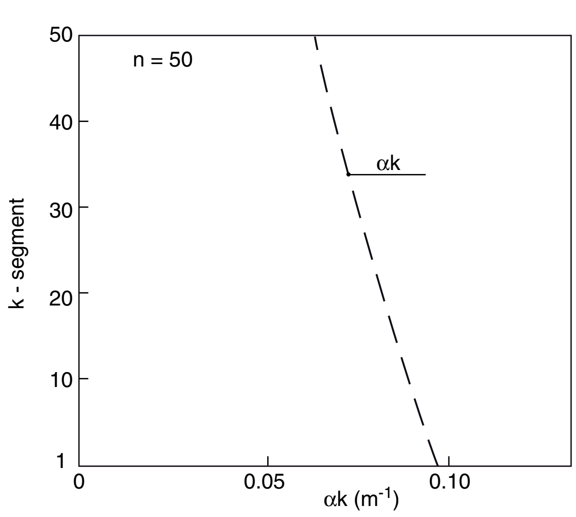

Figure 7

Coefficient of beam mode argument.

The coordinates of the cable and beam nodes, x_k and x'_k respectively, are listed in Table 3 and shown in Figure 5. Due to the sliding of the beam segments with respect to the cable ones, the values of x'_k are increased. The variation of its boundary values is zero as consequence of the fixed ends. The cable z_k - coordinate (23) varies linearly from the bottom to the top, Figure 6. The difference between the top and the bottom values reads $z_{n+1} - z_1 = n\pi$ which is in accordance with (19).

The FEM analysis of beam vibration shows that the top value of the mode envelope reads $C_t = 0.975$ and according to (76) one finds $\zeta_b = \zeta_1 = 3024.28$. The ζ coordinate of the vibration nodes by employing (74) is determined in Table 3 and shown in Figure 6. The diagrams of z_k and ζ_k are parallel straight lines. A considerably increased value of ζ_k compared to z_k is the result of flexural stiffness.

TABLE 3

Node coordinates, Mode 50

Cable ($EI = 0$): $\omega_{50} = 4.05447$ s^{-1} Eq. (20)			Beam ($EI > 0$): $\omega'_{50} = 5.47942$ s^{-1} Eq. (68)		
Segment k	x_k [m] Eq. (26)	z_k Eq. (23)	x'_k [m] Eq. (83)	$\zeta_k - \zeta_1$ ($z_k - z_1$) Section 3	ζ_k Section 3
51	2000.00	224.85	2000.00	50 π	3181.36
41	1428.26	193.47	1527.81	40 π	3149.94
31	942.40	162.05	1094.56	30 π	3118.53
21	542.40	130.64	697.69	20 π	3087.11
11	228.13	99.22	334.02	10 π	3055.69
1	0.00	67.81	0.00	0	3024.28

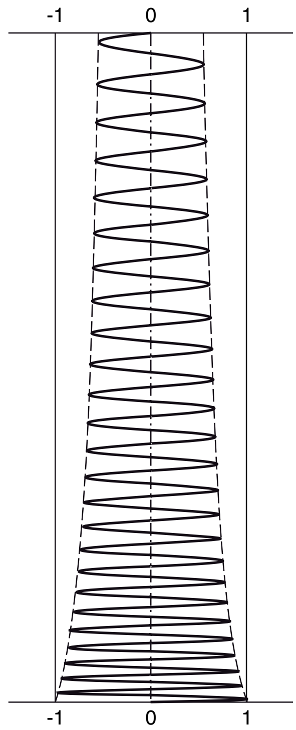

Figure 8

The 50-th natural mode of cable vibration.

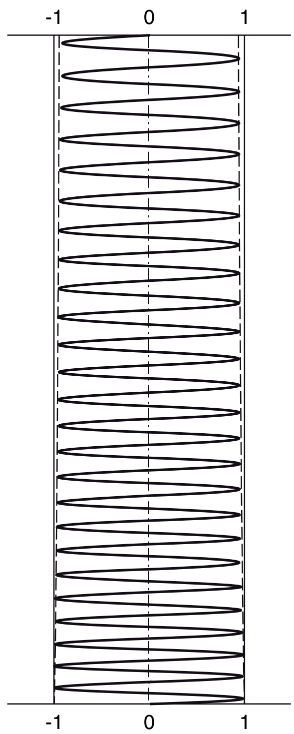

Figure 9

The 50-th natural mode of beam vibration.

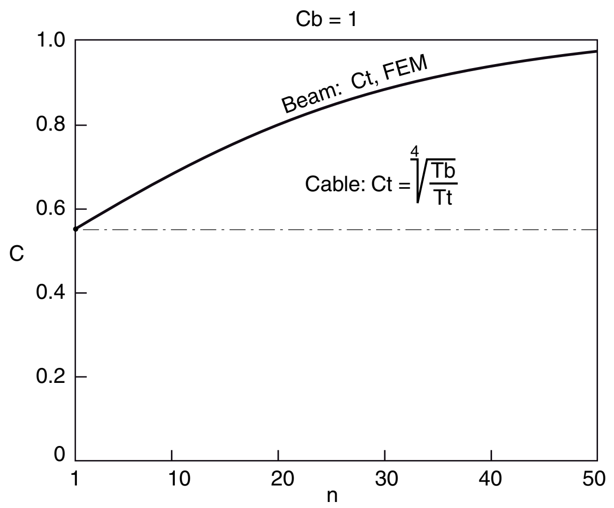

Figure 10

Top value of mode envelope.

The coordinate coefficient $\alpha_k = \pi/l'_k$, Equation (74), is shown in Figure 7. It is proportional to $\sqrt{Q_k}$ in accordance with (69).

The cable and beam vibration modes determined by the asymptotic integration and segmentation method are shown in Figures 8 and 9 respectively. The difference between the mode shapes is evident. In the latter case the mode envelope is expanding, and the bottom and top segment lengths are increasing and decreasing respectively.

The variation of the top value of the beam mode envelope, determined by the FEM analysis as function of mode number, is shown in Figure 10. It starts from the constant cable value, and approaches the unity when n increases to infinity.

CONCLUSION

Analysis of the natural vibration of tensioned risers is an important step in the investigation of vortex-induced forced vibration. The weak influence of the riser flexural stiffness is analysed through correlation of cable and beam vibration. The analytical procedures are employed for consistent analysis of the riser dynamic behaviour. The differential equation of cable vibration is solved exactly by Bessel's functions and approximately by the asymptotic integration. The problem of beam vibration is solved by the modification of the cable solution and its verification is performed by the finite element method. In both cases a fairly good agreement between the obtained results is achieved.

Based on the performed analyses the following conclusions may be drawn:

- the cable natural frequencies ω_n are linearly dependent on mode number n,
- the cable mode wave envelope depends only on distribution of tension force T_x and is the same for all vibration modes,
- the cable mode half-wave lengths, l_k, and square root of equivalent tension force, $\sqrt{T_{e_k}}$, are linearly dependent on half-wave number k,
- the equivalent tension force T_e transfers the variable coefficient in the differential equation into a constant value making in such a way its analytical solution is possible,
- the segmentation method for beam vibration analysis, based on the modification of the cable results, is a very effective iteration procedure,
- the beam natural frequencies are increased with respect to the cable ones due to flexural stiffness, but the increase is fairly low for the first modes,
- the position of the beam vibration nodes compared to those in the cable vibration is changed; mode half-wave lengths are increased and decreased at the bottom and top part of the riser, respectively,
- the beam mode wave envelope depends on the mode number; it changes from the cable envelope shape for $n = 1$ to constant value as n increases to infinity.

The obtained analytical results may be used only for risers of uniformly distributed characteristics. For risers of non-uniform characteristics, numerical methods, such as the finite element method, have to be used. In any case the analytical solution is useful for the explanation of the riser dynamic behaviour.

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APPENDIX

ITERATION PROCEDURE FOR BEAM VIBRATION ANALYSIS

Since all beam vibration parameters defined in Section 2 are non-linearly mutually dependent, the following iteration procedure has been developed for determining their values. For the initial values of the beam characteristics the available cable characteristics are used. The physical meaning of the iteration procedure is to approach from point C (cable) to point B (beam) in Figure 2 in a few steps.

The necessary initial beam characteristics taken from the cable analysis are the following:

– node coordinates: $x_k^{(0)} = x_k$, $k = 1,2...n+1$, Eq. (26)
– segment lengths: $l_k^{(0)} = l_k$, $k = 1,2...n$, Eq. (27)
– nodal forces: $T_k^{*(0)} = T_k$, $k = 1,2...n+1$, Eq. (2)
– equivalent forces: $T_{ek}^{*(0)} = T_{ek}$, $k = 1,2...n$, Eq. (42)

Initial values of the remaining beam characteristics:

– tension force due to bending, Equation (69):

$$Q_k^{(0)} = \left[\frac{\pi}{l_k^{(0)}}\right]^2 EI \ , \quad k = 1,2...n \tag{79}$$

– equivalent tension force, Equation (61):

$$T_{e_k}'^{(0)} = T_{e_k}^{*(0)} + Q_k^{(0)} \ , \quad k = 1,2...n \tag{80}$$

– natural frequency, Equation (66):

$$\omega_n'^{(0)} = \frac{\pi}{L\sqrt{m}} \sum_{k=1}^{n} \sqrt{T_{e_k}'^{(0)}} \tag{81}$$

The algorithm for the calculation of the beam vibration characteristics in the first and higher iteration steps, $i = 1, 2...$, is listed as follows:

– segment length, Equation (68):

$$l_k'^{(i)} = \frac{L\sqrt{T_{e_k}'^{(i-1)}}}{\sum_{k=1}^{n} \sqrt{T_{e_k}'^{(i-1)}}} \ , \quad k = 1,2..n \tag{82}$$

– node coordinates:

$$x_1^{(i)} = 0, \quad x_{k+1}^{(i)} = \sum_{j=1}^{k} l_j'^{(i)} \ , \quad k = 1,2...n \tag{83}$$

– cable tension forces in beam nodes, Equation (2):

$$T_k^{*(i)} = T_b + w\,x_k^{(i)} \ , \quad k = 1,2..n+1 \tag{84}$$

– equivalent cable force for beam segment, Equation (42):

$$T_{e_k}^{*(i)} = \frac{1}{4}\left[\sqrt{T_{k+1}^{*(i)}} + \sqrt{T_k^{*(i)}}\right]^2 \ , \quad k = 1,2...n \tag{85}$$

– tension force due to bending, Equation (69):

$$Q_k^{(i)} = \left[\frac{\pi}{l_k^{(i)}}\right]^2 EI \ , \quad k = 1,2...n \tag{86}$$

– equivalent beam tension force:

$$T_{e_k}'^{(i)} = T_{e_k}^{*(i)} + Q_k^{(i)} \ , \quad k = 1,2...n \tag{87}$$

– natural frequency, Equation (66):

$$\omega_n'^{(i)} = \frac{\pi}{L\sqrt{m}} \sum_{k=1}^{n} \sqrt{T_{e_k}'^{(i)}} \tag{88}$$

The iteration is performed for the n-th mode shape until the value of ω_n' is stabilised, *i.e.*:

$$\frac{\omega_n'^{(i)} - \omega_n'^{(i-1)}}{\omega_n'^{(i-1)}} \leq \varepsilon \tag{89}$$

where ε is a tolerable discrepancy.