ON SOME ASPECTS OF THE CONTINUUM-MECHANICS CONTEXT

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The Geomechanics Project[1]

À PROPOS DE CERTAINS ASPECTS DE LA MÉCANIQUE DU CONTINUUM

Ce document présente et discute de certains concepts de base de la mécanique du continuum de façon à permettre d'envisager des applications dans le domaine de la géophysique. Il aborde en particulier les concepts tels que les milieux de Cosserat, la répartition continue d'hétérogénéités et le rôle des symétries matérielles. Dans l'esprit de notre communication orale, la présentation garde un style informel.

ON SOME ASPECTS OF THE CONTINUUM-MECHANICS CONTEXT

In this paper certain basic concepts of continuum mechanics are presented and discussed in a manner that may be suggestive for applications in geophysics. In particular, ideas such as Cosserat media, continuous distribution of inhomogeneities, and the role of material symmetries are emphasized. In the spirit of a live meeting, the style of the presentation has been kept informal.

A PROPÓSITO DE CIERTOS ASPECTOS DE LA MECÁNICA DEL CONTINUUM

En este documento se presentan y se ponen en discusión ciertos conceptos básicos de la mecánica del continuum con objeto de permitir contemplar diversas aplicaciones en el campo de la geofísica. Básicamente, se entra en materia respecto a los conceptos tales como el soporte Cosserat, la distribución continua de carencia de homogeneidades y el cometido de las simetrías materiales. Permaneciendo en el entendimiento de una reunión dinámica, la presentación conserva siempre un estilo informal.

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* What follows is a written version of the presentation addressed by one of us (M.E.) to the *Eighth International Workshop on Seismic Anisotropy*. In thanking the organizers of this inspiring event for their invitation to prepare this written account, we decided to preserve the freshness and the lively atmosphere of the workshop by adhering to an almost verbatim transcription of the lecture, including the reproduction of the overhead transparencies used therein.

INTRODUCTION

The *Geomechanics Project* has been established this year as a joint venture between four companies and the *University of Calgary*. The first action taken by the consortium has been to establish an industrial research professor position at the *University of Calgary*, a position now occupied by Dr. Michael Slawinski, who was instrumental in getting the consortium started in the first place. Curiously, perhaps, the partnership was implemented with the *Department of Mechanical Engineering*, where I act as the University representative. So, it is this project that has brought a geophysicist (Michael) and a professor of mechanical engineering (myself) together, hopefully for the benefit of the sponsoring companies and beyond. One of our first collaborative efforts pertains to raytracing in anisotropic media following some ideas based on the calculus of variations and the application of Noether's theorem. We had initially thought of presenting these results here. This would have been doubly appropriate: firstly, because of the extraordinarily receptive and expert audience gathered here in this truly impressive conference; and, secondly, perhaps less importantly, because Pierre de Fermat, the father of the modern principle of stationary time, lived just next door, in Toulouse. Michael and I, in fact, went to Toulouse to pay our respects to Fermat, and we had a chance to see his likeness in sculpture at the museum. My surprise, however, was the beautiful arcade across from the Capitole, where the modern painter Moretti has depicted some of the famous sons and daughters of Toulouse. Sure enough, Fermat was there (Slide # 1); but just two tableaux further down, I recognized the likeness of Carlos Gardel (Slide # 2), the tango idol, who is practically a demi-god in my city of birth, Buenos Aires. It makes one wonder ...

As the case may be, Michael suggested that, instead of presenting our concrete results, perhaps it would be better that in this, my maiden speech in front of a

Slide # 1

Slide # 2

mainly geophysical audience, it would be interesting to offer an overview of how some of the work intensely focused on the application at hand fits in a wider continuum mechanics context. Whether or not this more vague and general talk has been a good idea, will be seen in the next few minutes, but, nervous as I am, it is too late to back off.

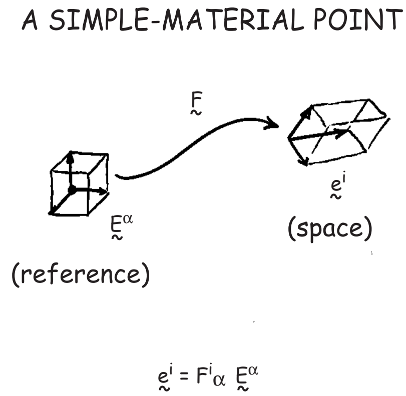
1 THE BODY MANIFOLD

What I would like to do today is to refer to some of the terminology used to establish the conceptual commonality between all pursuits dealing with media which, for better or worse, have been assumed *ab initio* to be continuous. The mathematical object corresponding to such physical entities is the *differentiable manifold*. In applications dealing with highly non-linear phenomena, sooner or later the differential-geometric technicalities involved in such objects will make their appearance; but not today, I promise. We have heard, in quite a few talks so far during this meeting (such as those of Drs. Sayers, Chesnokov, Bayuk, Carcione, and others) of very interesting ways to obtain "smeared out" continuous properties for the underlying continuum. Once this most important step is taken, the machinery of continuum mechanics can be brought to bear.

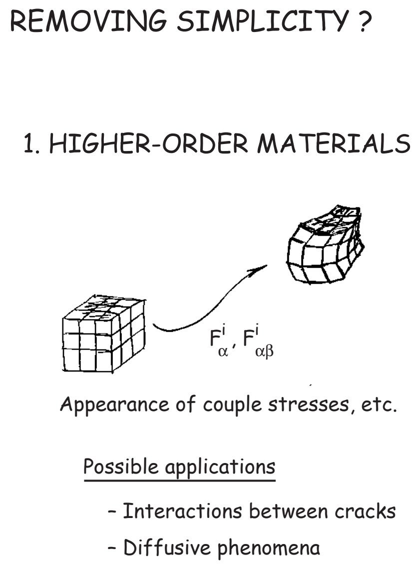
2 THE SIMPLE MATERIAL POINT

The first assumption implicit in most treatments is that the material is *simple*. What does that mean, and what possible practical repercussions could the replacement of that assumption by other ones have? To answer the first question, as to the nature of a simple material, we can intuitively say that it is the most *localized* non-trivial material one could possibly imagine. The response of such a material is completely characterized at each point by following the history of an infinitesimal parallelepiped around the point as it deforms into other infinitesimal parallelepipeds (Slide # 3). Accordingly, the transformation can be completely described by a tensor $\boldsymbol{F}$. Why a tensor? The answer is: necessarily so, by definition, since a tensor is precisely nothing more, or less, than a linear transformation between vector spaces. Already Cauchy, in his famous 1827 paper [1], understood it this way, although it took a while for the notion of a tensor to be formalized mathematically. Notice here that, if one chooses a basis $\boldsymbol{E}^{\alpha}$ (α = 1, 2, 3) in the reference parallelepiped, then $\boldsymbol{F}$ will transform it into the corresponding basis e^i (i = 1, 2, 3) in the "deformed" parallelepiped.

As to the second question, namely, what possible advantages could be expected from dropping the assumption of simplicity, there are many different avenues of approach, all of them implying some kind of "non-locality", action at a distance, or internal structure. One possible way to follow could be the route of *higher-order materials* (Slide # 4). A second-order material, for example, could be intuitively seen as characterized not just by a little parallelepiped (mathematically, a *first*



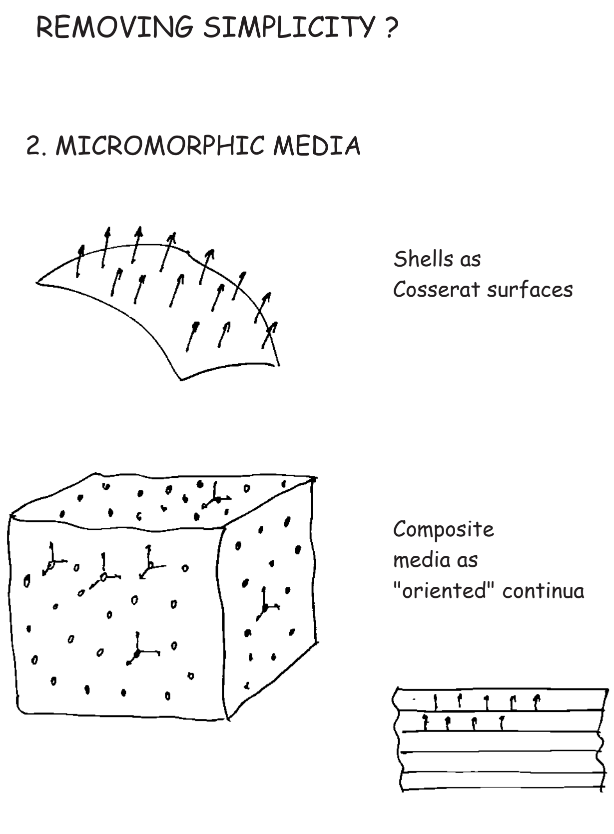
Slide # 3



Slide # 4

jet of deformations), but by a "larger" entity which can "feel" curvature-like effects (*second jets of deformations*). I have schematically indicated this with an array of 27 cells of an initially orthogonal grid surrounding the material point. One possible case in which this idea may be worth exploring was already mentioned by Dr. Sayers. It has to do with the consideration of interactions between cracks. These effects may require the introduction of second-order deformations, with their accompanying *couple-stresses*. Another example could emerge from a theory that includes diffusive effects, driven by density gradients.

There is, however, another possible way of replacing the simplicity assumption, which, I believe, may be of interest for some geophysical applications. This other way has to do with the representation of materials with internal structure, or *micromorphic media* (Slide # 5) [2]. When in a granular material one replaces the matrix-cum-granula complex by a homogenized single medium, the information relating to the grain deformations themselves is sadly lost in the anonymity of the averaging process. Thus, for example, high-frequency modes of vibration associated with the elasticity and inertia of the grains, become irretrievable. In the first decade of this century, the French brothers E. and F. Cosserat [3] proposed theories of deformable bodies in which the kinematics is modified to accommodate extra continuous degrees of freedom, made to correspond to the physical situation at hand. Shell theory, for instance, can be kinematically conceived as the deformation of a surface (two-dimensional manifold) accompanied by the deformation of a transverse vector field (or *director* field) characterizing the missing thickness. In granular materials, in addition to the underlying continuum, one needs at each point a deforming triad, whose elasticity and inertia come from the grain properties. Mathematically, the Cosserat brothers had unknowingly invented the concept of *principal fibre-bundle* and anticipated some of the fundamental geometrical work of Elie Cartan, particularly the concept of *repère mobile* [4]. A conceivable way to use these ideas in a stratified medium is to introduce a vector field representing the normal direction to the stratification and some measure of the layer deformation. This will, of course, engender some non-standard wave types. Treatments of this kind are common in theories of elastic dielectrics, ferromagnets, etc.



Slide # 5

3 ELASTICITY

Returning henceforth to a simple material point, it may so happen that all we need to characterize its material response is just the *present* value of the tensor $\boldsymbol{F}$: no history, no time rates, just the present value. In that case, we say that the material point is *elastic.* More particularly, a material point is said to be *hyperelastic* if it has a *stored energy* density function, W, so that the stress can be obtained from it as a potential (Slide # 6). Notice that this definition is completely general and, in particular, does not imply any smallness-of-strain assumption. It may be worthwhile to digress for a few moments on non-elastic materials. It has been suggested that perhaps the most general type of (simple) material can be characterized by a *constitutive law* of the form $\mathcal{F}\,[\boldsymbol{F}, \boldsymbol{T}, ...]$, where $\mathcal{F}$ is a *functional* of the whole past history of the deformation, the temperature, and possibly other variables (Slide # 7). This form of the constitutive law includes, in particular, short-memory effects such as those characterizing the

(HYPER-) ELASTIC MATERIAL POINT

$$W = W(\tilde{F})$$

elastic energy per unit
reference volume

$$\tilde{T} = \frac{\partial W}{\partial \tilde{F}}$$

Piola-Kirchhoff stress

Slide # 6

THE MOST GENERAL CONSTITUTIVE LAW ?

$$\left\{\begin{matrix} \text{stress} \\ \text{heat flux} \\ \vdots \end{matrix}\right\} = \mathop{F}_{-\infty}^{t} \left[\text{deformation, temperature, ...}\right]$$

→ General as this looks, this statement
is still controversial !

Slide # 7

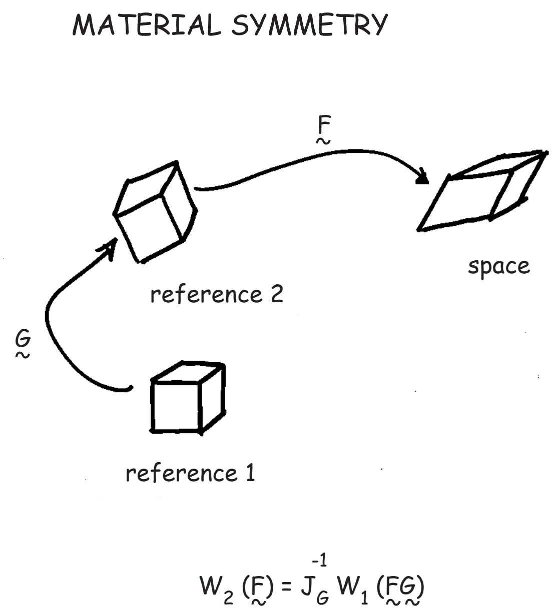
ordinary theory of viscoelasticity, but it can obviously accommodate much more general memory effects. Surprisingly, it is not clear whether or not this apparently most general constitutive law, consistent with the axioms of *causality and determinism* can accommodate all existing theories of material behaviour (most prominently, the theory of plasticity). Moreover, constitutive theory is still fraught with some pitfalls arising from a not yet completely understood continuum thermodynamical theory. If one adds to this scheme the uncertainties of field-matter interactions, it would be hard to disagree with Truesdell and Noll [5], who, more than thirty years ago, declared constitutive theory to be the main open question of continuum mechanics. Considerable progress has taken place since, but this is likely to remain an open challenge, perhaps due to the continuity assumption itself. In most applications, however, such philosophical niceties are usually hidden under the morass of necessary simplifying assumptions entailed in modeling real materials.

4 MATERIAL SYMMETRY

Returning to hyperelasticity, the potential function W may enjoy certain symmetries. In everyday terms, this means the following: if I am in the middle of an experiment designed to measure W as a function of $\boldsymbol{F}$, and the telephone rings in the next room and I go to answer, and if while I am gone a mischievous imp (just as Descartes' *mauvais génie* ([6], p. 413) effects a transformation $\boldsymbol{G}$ of some kind, by tampering with the specimen, and when I come back and continue with the experiment I get the same results (for all values of $\boldsymbol{F}$) as I would have got otherwise, then the tampering operation $\boldsymbol{G}$ is called a *material symmetry*. More precisely (Slide # 8), a *symmetry is a change of reference undetectable by any mechanical experiment.* Normally, under a change of reference given by a tensor $\boldsymbol{G}$ (representing the transformation from one reference parallelepiped to another) the stored energy function changes according to the formula:

$$W_2(\boldsymbol{F}) = J_G^{-1} W_1(\boldsymbol{F}\,\boldsymbol{G}) \qquad (1)$$

This formula is not difficult to understand. The subscripts 1 and 2 denote the two different references. The tensor multiplication in the argument of W_1 is the mathematical way of saying that if a given deformation $\boldsymbol{F}$ produces a certain value of the energy W_2, then in order to obtain the same value starting from the reference # 1, we should first apply the transformation $\boldsymbol{G}$ and only then apply $\boldsymbol{F}$. The reason for the appearance of the determinant J_G of $\boldsymbol{G}$, is that we have agreed to measure energy per unit volume of the reference parallelepiped. Now (Slide # 9), if $\boldsymbol{G}$ happens to be a material symmetry, the identity (1) should be satisfied with *the same* energy function, even though we have changed the reference. That is precisely what a symmetry means. In addition, for physical reasons, it is assumed that all symmetries are volume preserving, that is, they have a unit determinant. Therefore, we



Slide # 8

Consider a unimodular ($J_G = 1$) change of reference G. If it so happens that:

$$W_2(F) = W_1(F),$$

namely

$$W_1(FG) = W_1(F) \quad \text{for all } F$$

then G is a material symmetry. All material symmetries of a given point form a group $\mathcal{G}$

Slide # 9

conclude that $\boldsymbol{G}$ is a symmetry of the hyperelastic constitutive equation W, if the following identity is satisfied:

$$W(\boldsymbol{F}) = W(\boldsymbol{F}\,\boldsymbol{G}) \tag{2}$$

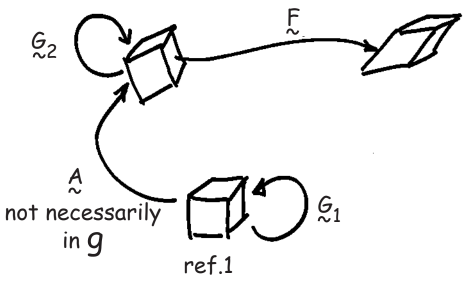
for all (non-singular) deformations $\boldsymbol{F}$.

All material symmetries of a given constitutive equation form a multiplicative group, $\boldsymbol{G}$, called the *material symmetry group*, which, by construction, is a subgroup of the unimodular group (the group of all matrices having a unit determinant). The symmetry group depends, naturally, on the reference chosen. Nevertheless (Slide # 10), this dependence is not arbitrary. Indeed, if the reference undergoes a transformation $\boldsymbol{A}$, then every symmetry $\boldsymbol{G}$ is transformed into the symmetry $\boldsymbol{A}\,\boldsymbol{G}\,\boldsymbol{A}^{-1}$. In other words, the two symmetry groups are *conjugate* of each other. This remark is of relevance to the so-called stress-induced anisotropy.

5 SOLIDITY AND ANISOTROPY

We are now in a position to define what is meant by an *elastic solid*. Intuitively, a solid has a preferred state in which it is “undistorted”, in the sense that no external forces are needed to maintain it in such a state. Moreover, if the solid has any symmetries at all, at the undistorted state these are bound to be rigid rotations and reflections only. Mathematically, therefore, we say that a simple elastic material point is a solid if there exists a reference for which its symmetry group is a subgroup of the orthogonal group (Slide # 11). The two extreme cases are given by: the trivial (*or triclinic*) case, in which the symmetry group consists of the identity (and its negative), and the fully isotropic case, in which the symmetry group coincides with the full orthogonal group. All other cases of anisotropy fall in between these two extremes It is important to note, however, that there is no *a priori* connection with the crystal classes. In fact, if the material is non-linearly elastic, in principle *any* subgroup of the orthogonal group could be a symmetry group, whether or not it corresponds to a crystal class! Consider, however, a material whose energy function happens to be completely defined by a symmetric fourth-order tensor, such as is the case in linear elasticity theory. Then, it can be shown that only a few subgroups are possible, and that they are all contained within the crystal classes. Notice that to prove this fact one does not need (nor

Note: the symmetry group depends in a "trivial" manner on the reference chosen.



$$W_2(\tilde{F}) = J_A^{-1}\, W_1(\tilde{F}\tilde{A}) = J_A^{-1}\, W_1(\tilde{F}\tilde{A}\tilde{G}_1)$$

$$= W_2(\tilde{F}\underbrace{\tilde{A}\tilde{G}_1\tilde{A}^{-1}}_{\tilde{G}_2})$$

Slide # 10

SOLIDITY

A material point is a solid point if there exists a reference for which

$g \subset o$

(orthogonal group)

No a priori connection with the crystal classes !

Slide # 11

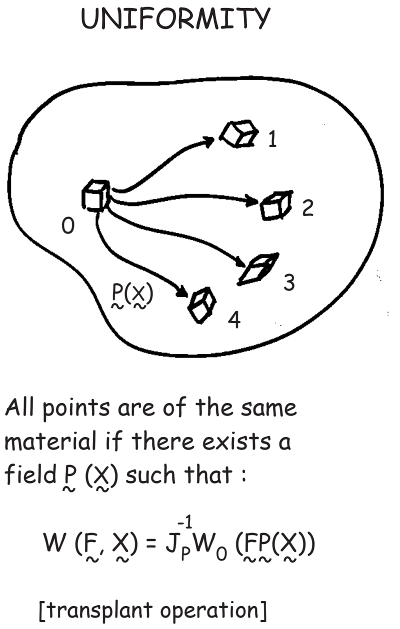
would it be correct) to look at the crystal classes of, say, the material components of a composite. A proof of this theorem, mentioned in Prof. Helbig's book on seismic anisotropy [7], was provided a few years ago by Huo and del Piero [8]. A final remark: for non-simple (even if elastic) materials, the symmetry group choice is a much larger set and, in mundane terms, a completely different animal (the composition law, for instance, is not just a simple multiplication of matrices).

6 UNIFORMITY AND HOMOGENEITY

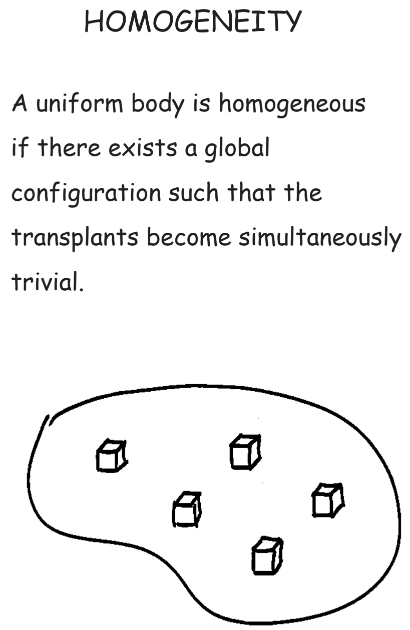
We finally come to a much more complex issue, namely, that of homogeneity. The formalization of this concept was achieved through the pioneering work of Kondo [9], Bilby [10], Kroener [11], and others, who were trying to model continuous distributions of dislocations. Their modus operandi, therefore, was based on looking at the geometrical pictures of dislocated crystalline arrays, identifying the corresponding Burger's circuits, and then effecting a heuristic passage to the limit as the size of the lattice unit is allowed to go to zero while retaining the main geometric features of the dislocation. This ingenious procedure, supplemented by a good dose of differential-geometric virtuosity, permitted them to define dislocation densities by means of geometric entities such as the Cartan torsion of a connection. A different avenue of approach, leading to similar, if not identical, results, was propounded by Noll [12] and Wang [13], who developed the theory without recourse to the underlying crystalline structure. This can be achieved as follows (Slide # 12): starting from a given elastic constitutive equation for a whole body, $W = W(\boldsymbol{F}, \boldsymbol{X})$, in principle the dependence on position $\boldsymbol{X}$ in the body can be completely arbitrary, even to the extent that the material itself may be physically different from point to point. At this point one may be tempted to say: if the constitutive function is independent of $\boldsymbol{X}$, then, and only then, we may be sure that the material is the same at all points. But this would be incorrect, because it may so happen that the material is the same at all points but happens to be pointwise at different states. Mathematically, this would be recognized by the existence of a set of "transplant" operations $\boldsymbol{P}(\boldsymbol{X})$ which permit to identify the constitutive law at different points $\boldsymbol{X}$ with the constitutive law at a fixed point $\boldsymbol{X}_0$ via a linear transformation, as follows:

$$W(\boldsymbol{F}, \boldsymbol{X}) = \boldsymbol{J}_{\boldsymbol{P}}^{-1}\, W(\boldsymbol{F}\,\boldsymbol{P}(\boldsymbol{X}), \boldsymbol{X}_0) \qquad (1)$$

which is obtained by a slight reinterpretation of the physical meaning of equation (1). If such a field of transplants exists, we say that the body is *uniform*, namely, is made of the same physical material at all points. What is then *homogeneity*? It is simply a particular case of uniformity, namely, the case in which it is possible to bring the body to a global configuration such that the transplants are not necessary (that is, they are equal to the identity tensor). In that case (Slide # 13), all the material points are arranged in the

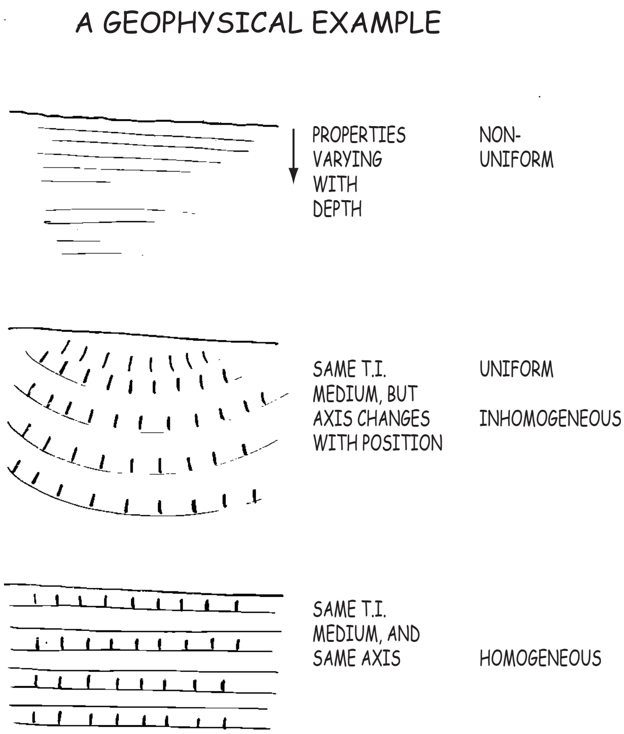


Slide # 12



Slide # 13

body in a nice way, and the constitutive equation indeed does not depend on position. Let us consider some simple geophysical examples (Slide # 14). If we have a continuously stratified medium in which the material properties vary with depth, the medium is non-uniform: it is simply not the same material at all depths. Consider now a medium which is made of the same transversely isotropic material at all points, but let the stratification be such that the axes of anisotropy are not parallel to each other. We then have a case of uniformity, but with inhomogeneities. In the terminology of dislocations, we have a medium with a continuous distributions of dislocations (it cannot be assembled by fitting together identical rectangular cells). Finally, if in the previous example the axes of anisotropy happen to be parallel (or if they could be brought to such a state by a smooth deformation while removing all internal stresses), then we have a case of homogeneity. Notice that in the second example, the constitutive equation will depend explicitly on position, although the material is the same at all points. The technical treatment of the theory of inhomogeneities is, unfortunately, not elementary. The different orientations of the axes of anisotropy in our example give rise to a non-integrable distant parallelism, with a non-vanishing Cartan torsion (Slide # 15). Wave fronts will be curved accordingly.



Slide # 14

THESE TRANSPLANT OPERATIONS
CREATE A DISTANT PARALLELISM
(vectors at different points are "materially"
parallel if they have the same components
in the induced bases). THIS
PARALLELISM, IN GENERAL, HAS A
NON-ZERO TORSION TENSOR T

But note : - the transplants are essentially not unique for a continuous symmetry group (such as T.I.)

- the parallelism may not be global (use G-structures, etc.)

IF A MATERIAL PARALLELISM CAN BE FOUND
SUCH THAT $T = 0$, WE HAVE HOMOGENEITY
(THE PARALLELEPIPEDS CAN BE STRAIGHTENED BY
A SINGLE CHANGE OF CONFIGURATION OF THE BODY)

Slide # 15

CLOSING REMARKS

I have reached the end of the allotted time and, probably, the end of your patience. It is my conviction that between the two extremes of research driven solely by the need of immediate applicability, on the one hand, and that driven by mere curiosity and search for elegance, on the other hand, there lies a fertile field of investigation driven by the harmonious coexistence of both. Mechanics is an old and resilient science. It can take, and has taken, apparent abuse from all camps, but has always emerged victorious and enriched. Every rediscovery has contributed a new perspective, every application to a new domain has revealed unimaginable vistas. Today, I have touched upon a few topics which may very well be irrelevant to exploration geophysics, and I would be the last person to advocate that such topics become a part of geophysical research. Nevertheless, it is quite possible that some of the ideas that have been developed under different motivational experiences may inspire fresh points of view for geophysical applications.

ACKNOWLEDGMENTS

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