FREQUENCY DEPENDENCE OF PHYSICAL PARAMETERS OF MICROINHOMOGENEOUS MEDIA. SPACE STATISTICS

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DÉPENDANCE EN FRÉQUENCE DES PARAMÈTRES PHYSIQUES DE MILIEUX MICROHÉTÉROGÈNES. STATISTIQUES SPATIALES

La technique par diagrammes appliquée au calcul des propriétés dynamiques d'un milieu anisotrope ayant une distribution aléatoire d'inclusions (pores, fissures) est ici développée. La description statistique des inclusions est déterminée par une fonction de distribution reposant sur cinq groupes de paramètres :

– les coordonnées,
– les angles d'orientation des formes,
– les angles d'orientation des axes cristallographiques,
– les rapports de forme (dans le cas d'inclusions de forme ellipsoïdale),
– les types de phase d'inclusions.

Une telle approche statistique permet de prendre en compte tout type et tout ordre d'interaction de corrélation entre les inclusions. La série de diagrammes est construite pour une fonction de Green moyenne (GF). La sommation précise de cette série donne une équation dynamique non linéaire pour une GF moyenne (équation de Dyson). Le noyau de cette équation est un opérateur de masse qui dépend de la fréquence et peut être représenté sous forme de série de diagrammes sur une fonction précise GF. L'opérateur de masse coïncide avec le tenseur complexe réel d'élasticité (ou de conductivité) dans une approximation locale. On obtient un développement du tenseur élastique (de transport) dynamique effectif par rapport aux fonctions de distribution à tout ordre. Il est montré que la corrélation entre les homogénéités peut produire une anisotropie des paramètres effectifs élastiques et de transport. Dans l'approche de la corrélation, l'influence de la dispersion sur les constantes élastiques effectives est étudiée. La dépendance en fréquence d'un coefficient d'anisotropie élastique en fonction de la statistique de distribution spatiale des inclusions (matrice isotrope et inclusions isotropes sphériques) est ainsi obtenue.

FREQUENCY DEPENDENCE OF PHYSICAL PARAMETERS OF MICROINHOMOGENEOUS MEDIA. SPACE STATISTICS

The diagram technique for calculation of the dynamic properties of an anisotropic media with randomly distributed inclusions (pores, cracks) is developed. Statistical description of inclusions is determined by distribution function dependent on five groups of parameters:

– over coordinates;
– over angles of orientation of shapes;

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– over angles of orientation of crystallographic axes;
– over aspect ratio (in a case of ellipsoidal inclusions);
– over types of phase of inclusions.

Such statistical approach allows to take into consideration any type and order of correlation interactions between inclusions. The diagram series for an average Green function is (GF) constructed. The accurate summation of this series leads to a nonlinear dynamic equation for an average GF (Dyson equation). The kernel of this equation is a mass operator which depends on frequency and can be presented in a form of diagram series on accurate GF. The mass operator coincides with effective complex tensor of elasticity (or conductivity) in a local approximation. An expansion of effective dynamic elastic (transport) tensor on distribution functions of any order is obtained. It is shown that correlation between homogeneities can produce an effective elastic and transport parameters anisotropy. In correlation approximation the dispersion dependencies of the effective elastic constants are studied. Frequency dependencies of a coefficient anisotropy of the elastic properties as function of statistical distributed inclusions over coordinates (isotropic matrix and isotropic (spherical) inclusions) are obtained.

DEPENDENCIA DE LOS PARÁMETROS FÍSICOS DE MEDIOS MICRO NO-HOMOGÉNEOS CON RESPECTO A LA FRECUENCIA. ESTADÍSTICA ESPACIAL

Se desarrolla la técnica de diagramas para el cálculo de las propiedades dinámicas de un medio anisotrópico con inclusiones aleatoriamente distribuidas (poros, grietas). La descripción estadística de las inclusiones se determina por función de distribución sobre cinco grupos de parámetros:

– sobre las coordenadas;
– sobre los ángulos de orientación de las formas ;
– sobre los ángulos de orientación de los ejes cristalográficos ;
– sobre la relación de aspecto (en caso de inclusiones elipsoidales) ;
– sobre los tipos de fase de las inclusiones.

Dicho enfoque estadístico permite tomar en consideración cualquier tipo y orden de interacciones de correlación entre inclusiones. Se construyen las series de diagramas para una función Green promedio (GF). La suma exacta de estas series lleva a una ecuación dinámica no lineal para una GF promedio (ecuación de Cyson). El núcleo de esta ecuación es un operador masivo que depende de la frecuencia y que puede ser presentado en forma de series de diagramas de GF exactas. El operador masivo coincide con un tensor complejo efectivo de elasticidad (o conductividad) en una aproximación local. Se obtiene una expansión del tensor elástico (de transporte) dinámico efectivo sobre las funciones de distribución de cualquier orden. Se demuestra que la correlación entre homogeneidades puede producir una anisotropía efectiva de los parámetros elásticos y de transporte. Se estudian en una aproximación de correlación la dependencia de la dispersión con respecto a las constantes elásticas efectivas. Se obtiene la dependencia de la frecuencia con respecto a un coeficiente de anisotropía de las propiedades elásticas como función de inclusiones estadísticamente distribuidas sobre coordenadas (matriz isotrópica e inclusiones isotrópicas (esféricas)).

INTRODUCTION

Determination of elastic constants of inhomogeneous anisotropic multicomponent media is based on effective procedure of averaging. In case, when size of inclusions (a) is much less then wavelength (L) or corresponding radius of correlation (r), statistical averaging over ensemble is equivalent to averaging over volume. Such averaging is not related to small fluctuations as usually use in the first Born approximation. In spite of this simplification the problem of averaging of wave equation is not trivial due to the large (although short-wavelength) fluctuations of elastic constants of the medium.

In formal algebraic sense the problem of averaging was solved by Willis (1977), Shermergor (1977), who wrote the equation for average Green function (Dyson equation). However, the kernel of this equation (in algebraic approach) is expressed via “freedom” (i.e. homogeneous matrix without inclusions) Green function and is valid for small fluctuations only.

In present paper this approach is generalized on to the case of large fluctuations. This goal can be reached by expressing the kernel of Dyson equation (mass operator) via Green function, which takes into account interaction between inclusions. This expression can be obtain by summing up all members of series of perturbation presented in diagram form.

In present paper the diagram technique for calculation of the dynamic properties of an anisotropic media with randomly distributed inclusions (pores, cracks) is developed. Statistical description of inclusions is determined by distribution function dependent on five groups of parameters:

– over coordinates;
– over angles of orientation of shapes;
– over angles of orientation of crystallographic axes;
– over aspect ratio (in a case of ellipsoidal inclusions);
– over types and sorts of phase of inclusions.

Such statistical approach allows to take into consideration any type and order of correlation interactions between inclusions. The accounting of correlations leads to generation of anisotropy of effective elastic properties. In correlation approximation the dispersion dependencies of the effective elastic constants are calculated. Frequency dependence of coefficient anisotropy of the elastic properties as a function of space statistically distributed inclusions is obtained.

1 DIAGRAM TECHNIQUE FOR DYNAMIC GREEN FUNCTION OF ELASTIC FIELD

Let's consider wave equation:

$$Lu = f \tag{1}$$

where operator L has a form:

$$L_{ik}(x) = \delta_{ik}\rho\frac{\partial^2}{\partial t^2} + \nabla_j C_{ijKl}(x)\nabla_l \tag{2}$$

and depends on parameters of media- density-ρ and tensor of elastic constants $C_{ijkl}(x)$.

The solution of the equation (1) can be written as:

$$u_i = \int G_{ik}(t-t_1, x-x_1)\, f_k(x_1, t_1)\, dx_1 dt_1 \tag{3}$$

Substitution (3) to (1) leads to equation for Green function G:

$$L_{ik}(x)\, G_{kj}(t-t_1, x-x_1) = -\delta_{ij}\delta(t-t_1)\,\delta(x-x_1) \tag{4}$$

We will consider the elastic constants of micro-inhomogeneous medium as a random field and correlation characteristics of this field will determine the average elastic field of displacement (3) for media with inclusions.

In this paper for simplicity we will neglect the fluctuation of density and will not consider its coordinate dependence.

It is obvious, that average displacement $<u>$ can be expressed via average Green function $<G>$:

$$\langle u_i(t,x)\rangle = \int \langle G_{ik}(t-t_1, x-x_1)\rangle\, f_k(t_1, x_1)\, dx_1 dt_1 \tag{5}$$

Let's carry out the motion equation for average Green function $<G>$. Elastic modules can be expressed as a sum average and random terms:

$$C_{ijkl}(x) = \langle C_{ijkl}(x)\rangle + C'(x) \tag{6}$$

and operator L, correspondently:

$$L_{ij}(x) = \langle L_{ij}(x)\rangle + L'_{ij}(x) \tag{7}$$

where:

$$\langle L_{ik}(x)\rangle = -\delta_{ik}\,\rho\frac{\partial^2}{\partial t^2} + \nabla_j \langle C_{ijkl}(x)\rangle \nabla_l$$
$$L'_{ik}(x) = \nabla_j C'_{ijkl}\nabla_l \tag{8}$$

Substituting (7) into (4) and introducing Green function $G^0_{ik}(t-t_1, x-x_1)$, which satisfy an equation with average elastic constants (without fluctuations) we can write:

$$\langle L_{ij}(x)\rangle\, G^0_{jk}(t-t_1, x-x_1) = \delta_{ik}\delta(t-t_1)\,\delta(x-x_1) \tag{9}$$

It is possible to show that exact Green function G, which takes into account all fluctuations satisfy an integral equation:

$$G_{ik}(t-t_1, x-x_1) = G^0_{ik}(t-t_1, x-x_1) + \int G^0_{il}(t-t_2, x-x_2)\, L'_{lp}(x_2)\, G_{pk}(t-t_1, x-x_1)\, dx_2 dt_2 \tag{10}$$

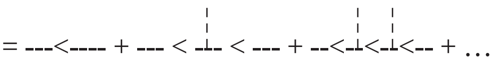
Let's denote:

$$G^0 = \text{----<-----};\ L' = \text{¦};\ \langle G\rangle \text{ ----- ◀ -----} \tag{11}$$

and express the iteration series for Equation (10):

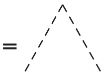
$$G = G^0 + G^0 L' G^0 + G^0 L' G^0 L' G^0 + \ldots \tag{12}$$

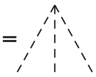
(for simplicity we do not write arguments, i.e. consider G^0, L' as the operators) in a graphic form:

G =  (13)

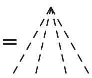
which is convenient for averaging.

Let's introduce the correlation functions of arbitrary order:

$K_{12} \equiv K^{nl}_{ij}(x_1, x_2) \equiv \langle L'_1 L'_2\rangle =$ 

$K_{123} \equiv K^{mn}_{ijkl} \equiv \langle L'_1 L'_2 L'_3\rangle =$ 

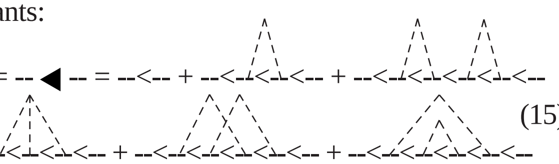
$K_{1234} \equiv K^{mnpq}_{ijkl}(x_1,x_2,x_3,x_4) \equiv \langle L'_1 L'_2 L'_3 L'_4\rangle - \langle L'_1 L'_2\rangle\langle L'_3 L'_4\rangle$

$- \langle L'_1 L'_3\rangle\langle L'_2 L'_4\rangle - \langle L'_1 L'_4\rangle\langle L'_2 L'_3\rangle =$  (14)

and etc.

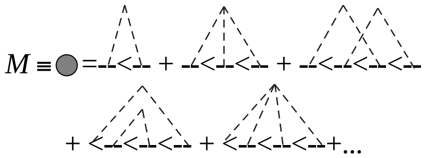
A correlation function of n-th order $K_{1,\ldots,n}$ can be obtained by subtracting from central moment of n-th order $\langle L'_1,\ldots, L'_n\rangle$ of all sorts of products of correlation functions of low orders.

Averaging of (13) and substituting (14), obtains a series for the average Green function $<G>$, which depends on fluctuations of the tensor of elastic constants:

$<G>$ =  (15)

2 DYSON EQUATION

Let's call the diagram in (15) nonreducible if it can not be derived into two similar diagrams of lower order by cutting along a solid line. There is only one in (15), (4-th in the right part of the Equation (15)) which satisfies this condition. Let's introduce the mass operator as a sum of nonreducible diagrams:



(16)

It is easy to see, that any reducible diagram in (15) is reduced to similar nonreducible diagram (16), connecting one, two and etc lines of G^0:

<G> = -- ◀ -- = --<-- + --<-●-<--
+ --<-●-<-●-<-- + ... (17)

After transformation of (17) we can write:

<G>= -- ◀ -- = --<-{--<- + -<-●-<-
+ --<-●-<-●- + ...} (18)

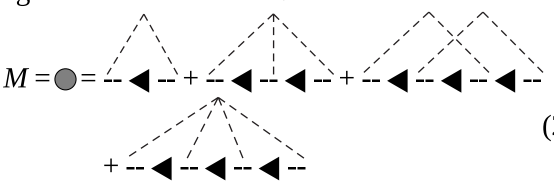
After summing up in brackets (18) with help of (17) we obtain Dyson equation in graphic form for average Green function:

<G>= -- ◀ -- = --<-- + -- -- ◀ -- (19)

or in analytical form:

$$\langle G_{ik}(t-t_1, x-x_1)\rangle = G^0_{ik}(t-t_1, x-x_1) + \int dt_2 dt_3 dx_2 dx_3 G^0_{ik}(t-t_2, x-x_2)\, M_{km}(t_2-t_3, x_2-x_3)\, \langle G_{mk}(t_3-t, x_3-x)\rangle \quad (20)$$

Using (17) the mass operator M can be written via average Green function $<G>$:



(21)

3 EFFECTIVE ELASTIC CONSTANTS OF MEDIA WITH INCLUSIONS. FREQUENCY DEPENDENCE

Let's consider the case, when the size of inclusions and distance between them (correlation radius r^{corr}) is much less than the wavelength. Then we can use the local approximation and write the Dyson equation (20) in a form:

$$\langle G_{ik}(t-t_1, x-x_1)\rangle = G^0_{ik}(t-t_1, x-x_1) + \int dt_2 dt_3 d_y G^0_{ik}(t-t_2, x, y)\, M^*_{km}(t_2-t_3, y)\, \langle G_{mk}(t_3-t, x, y)\rangle \quad (22)$$

where:

$$M^*_{ik}(t-t_1, x) = \int dy M_{ik}(t-t_1, x, y) \quad (23)$$

Since each diagram in (21) is begins and ends with L' line and contents of an operator of gradient, then (23) can be presented in a form:

$$M^*_{ik}(t-t_1, x) = \nabla_m \left[C^*_{imkn}(t-t_1) - \langle C^*_{imkn}\rangle\, \delta(t-t_1) \nabla_m \right] \quad (24)$$

Using equation (3) we now return to the differential equation for the average elastic field in a medium with inclusions:

$$\int dt_1\, L^*_{ik}(t-t_1, x) < u_k(t_1, x) > = f_i(t, x) \quad (25)$$

where:

$$L^*_{ik}(t-t_1, x) = -\delta_{ik}\, \rho \frac{\partial^2}{\partial t^2}\, \delta(t-t_1) + \nabla_j C^*_{ijkl}(t-t_1, x)\rangle \nabla_l \quad (26)$$

For calculating the effective tensor of elastic constants let's consider a correlation approximation for the mass operator:

$$M_{ik}(t-t_1, x-x_1) =$$ 

$$= K^{lm}_{ik}(x-x_1)\, G^0_{pm}(t-t_1, x-x_1) \quad (27)$$

where:

$$K^{pm}_{ik}(x-x_1) = \langle L'_1 L'_2 \rangle = \langle \nabla_j C'_{ijkl}(x)\, \nabla_l \nabla_q C'_{pqmn}(x) \nabla_n \rangle \quad (28)$$

From here we can write:

$$C^*_{ijkl}(t-t_1) = \langle C_{ijkl}\rangle\, \delta(t-t_1) + \int dx_3 G^0_{mn,pq}(t-t_1, x_2-x_3)\, B^{ijmn}_{pqkl}(x_2-x_3) \quad (29)$$

here:

$$G^0_{mn,pq}(t-t_1, x_2-x_3) = \nabla_p \nabla_q G^0_{mk}(t-t_1, x_2-x_3)$$
$$B^{ijmn}_{pqkl}(x_2-x_3) = \langle C'_{pqkl}(x_2)\, C'_{ijmn}(x_3)\rangle \quad (30)$$

In a general case the correlation functions of fluctuations of elastic tensor are defined by statistical distribution of inclusions with respect to the position (coordinate of center), crystallographic axes, orientation, aspect ratios and phases. Here for simplicity we will consider the case of space statistics of point inclusions of one sort in homogeneous matrix. Then:

$$\langle C_{ijkl}\rangle = C^M_{ijkl} + \tilde{C}^I_{ijkl}\, n v^I \quad (31)$$

where: C^M_{ijkl}, C^I_{jkl} -tensors of elastic constants of matrix (M) and inclusions (I), correspondingly; n-density of number of inclusions, v-volume of inclusion and $\tilde{C}^I_{ijkl} = C^I_{ijkl} - C^M_{ijkl}$. Value of correlation function can be presented in a form:

$$\langle B^{pqmn}_{ijkl}\ (x - x_1)\rangle = \tilde{C}^I_{ijkl} + \tilde{C}^I_{pqmn} (v^I)^2 g_{12} (x - x_1) \quad (32)$$

here: $g_{12}\ (x - x_1)$ – pair correlation function of location of inclusions.

Taking into account that that operator (26) in equation (25) depends on time difference and supposing that distribution of inclusions in space is statistically homogeneous, we can apply the Fourier transformation to equation (29). This procedure allows to obtain frequency dependent elastic constants:

$$C^*_{ijkl}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} C^*_{ijkl}(\tau) = \langle C_{ijkl}\rangle + \int \frac{dk}{(2\pi)^3} G^0_{mn,pq}(\omega, k) B^{ijmn}_{pqkl}(k) \quad (33)$$

here:

$$B^{ijmn}_{pqkl}(k) = \int dx e^{ik\,(x - x_1)} B^{ijmn}_{pqkl}(x - x_1)$$
$$G^0_{mn,pq}(\omega, k) = ik_p ik_q\ G^0_{mn}(\omega, k) \quad (34)$$
$$G^0_{mj}(\omega, k) = \frac{\delta_{ij} - v_i v_j}{\langle\mu\rangle k^2 - \rho\omega^2} + \frac{v_i v_j}{\langle\lambda + 2\mu\rangle k^2 - \rho\omega^2}$$

Last formula is a Fourier-image of Green function G^0 and $v_i = k_1/k$.

Lets consider far order in distribution of inclusions, i.e. periodical distribution with period $\lambda_0 = 2\pi/|k_0|$ << $\lambda = 2\pi/|k|$ in direction of $\vec{k}_0/k_0$ and amplitude of:

$$g_{12}\ (x_2 - x_3) = n^2\ g^{(1)} \cos k_0\ (x_2 - x_3)\ e^{-\alpha|x_2 - x_3|} \quad (35)$$

We assume that the attenuation α is small compared with the value of inverse wavelength ($1/\lambda$) and neglect it. Then the Fourier – image of g can be written in a form:

$$g_{12}(k) = \frac{1}{2}\ n^2 g^{(1)} \left[\delta\ (k - k_0) + \delta\ (k + k_0)\right] \quad (36)$$

Substituting (36) into (34) and (33) we obtain (37)

In a case of isotropic inclusions:

$$\tilde{C}^I_{ijkl} = \tilde{\lambda}^I\ \delta_{ij}\delta_{kl} + \tilde{\mu}^I\ (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \quad (38)$$

and hence:

$$\tilde{C}^I_{ijkk} \left(3\,\tilde{\lambda}^I + 2\,\tilde{\mu}^I\right) \delta_{ij} \cdot k_i k_j\ \tilde{C}^I_{ijkl} = k^2\ \tilde{\lambda}^I\ \delta_{k1} + 2\,\tilde{\mu}^I\ k_l k_k \quad (39)$$

Substituting (39) into (37), obtain (40) where $v^0_i \equiv k^0_i/|k|$ is a unit vector in the direction $\vec{k}_0$.

Rewrite (40) in a compact form: (41).

Substitution (34) and:

$$G^0_{nn} = \frac{2}{\langle\mu\rangle\ k_0^2 - \rho\omega^2} + \frac{1}{\langle\lambda + 2\mu\rangle\ k_0^2 - \rho\omega^2} \quad (42)$$

into (41), obtain a complete dependence of effective elastic parameters $C^*_{ijkl}(\omega)$ on frequency.

$$C^*_{ijkl}(\omega) = \langle C_{ijkl}\rangle + \frac{n^2 (v^I) g^{(1)}}{(2\pi)^3}\ G_{mn,pq}(\omega,\ k)\, \tilde{C}^I_{ijmn} \tilde{C}^I_{pqkl}$$
$$= \langle C_{ijkl}\rangle - \frac{n^2 (v^I) g^{(1)}}{(2\pi)^3 \langle\mu\rangle k_0^2 - \rho\omega^2} \cdot \left[\tilde{C}^I_{ijmn} k_p k_q \tilde{C}^I_{pqkl} - \frac{\langle\lambda + 2\mu\rangle}{\langle\lambda + 2\mu\rangle k_0^2 - \rho\omega^2} \cdot \tilde{C}^I_{ijmn} k^0_m k^0_n k^0_p k^0_q \tilde{C}^I_{pqkl}\right] \quad (37)$$

$$C^*_{ijkl}(\omega) = \langle C_{ijkl}\rangle - \frac{n^2 (v^I)^2 g^{(1)}}{(2\pi)^3} \cdot \frac{k_0^2}{\langle\mu\rangle k_0^2 - \rho\omega^2} \cdot \Bigg[(3\tilde{\lambda}^I + 2\tilde{\mu}^I)\tilde{\lambda}^I \delta_{ij}\delta_{kl} + 2(3\tilde{\lambda}^I + 2\tilde{\mu}^I)\tilde{\mu}^I \delta_{ij} v_i v_j$$
$$- \frac{\langle\lambda + \mu\rangle}{\langle\lambda + 2\mu\rangle}\left[(\tilde{\lambda}^I)^2 \delta_{ij} + 2\tilde{\lambda}^I \tilde{\mu}^I (\delta_{ij} v^0_k v^0_l + \delta_{kl} v^0_i v^0_j + 4(\tilde{\mu}^I)^2 v^0_i v^0_j v^0_k v^0_l\right]\Bigg] \quad (40)$$

$$C^*_{ijkl}(\omega) = \langle C_{ijkl}\rangle - \frac{n^2 (v^I)^2 g^{(1)}\ k_0^2}{(2\pi)^3} \left[\tilde{\lambda}^I\ G^0_{nn}\ \delta_{ij}\ +\ 2\ \tilde{\mu}^I\ G^0_{ij}\right] \cdot \left[\tilde{\lambda}^I\ \delta_{kl}\ +\ 2\ \tilde{\mu}^I\ v_k v_l\right] \quad (41)$$

Using (42), find out an expression of coefficient anisotropy as a function of frequency:

$$\alpha_P = \frac{\left|C_{11}^* - C_{33}^*\right|}{\langle C_{11}\rangle} = \frac{n^2 (v^I)^2 g^{(1)} k_0^2}{(2\pi)^3} \cdot \frac{2\tilde{\mu}^I}{\langle \lambda + 2\mu \rangle} \cdot \left[\frac{\tilde{\lambda}^I}{\langle \mu \rangle k_0^2 - \rho\omega^2} + \frac{2(\tilde{\lambda}^I + \tilde{\mu}^I)}{\langle \lambda + 2\mu \rangle k_0^2 - \rho\omega^2} \right] \quad (43)$$

As follows from (42), when frequency increase an effective elastic constants decrease. This effect leads to decrease of elastic velocities, when frequency increase. It is important to point out that in our case a value of $(1/k)$ is small compared with wavelength and while $\rho\omega^2$ is small compared with $\langle \mu \rangle\, k_0^2$. This means that denominator of the Green function does not vanish.

CONCLUSIONS

For microinhomogeneous media with inclusions the problem of construction of macroscopic dynamic equations is solved using the diagram technique.

The exact expressions for the frequency dependence of the tensor of elastic constants is deduced in a case of a spatial distribution of inhomogeneities.

Frequency dependence of elastic constants and coefficients of anisotropy is obtained in a frame of self-consistent scheme.

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